INVERSE STEADY HEAT CONDUCTION IN A TWO-DIMENSIONAL COMPOSITE MEDIA

التوصيل الحراري المستقر المعكوس في وسط مركب ثنائي الأبعاد

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ملخص:

في هذا البحث تم ايجاد طريقة تحليلية لحل نوع معين من مسائل التوصيل الحراري المستقر المعكوسة في حائط مركب ثنائي الأبعاد مستوي او اسطواني، وذلك بتطوير طرق (سبق لنا ايجادها) تستخدم في حل مسائل معكوسة لأشكال هندسية بسيطة ذات حائط مفرد (مستوى او اسطواني). هذا وقد تم عمل تطوير اخر في طريقة الحل المقترحة لكي تستخدم ايضاً في حل هذا النوع من مسائل انتقال الحرارة في حالة وجود توليد حراري في الحائط المركب، هذا ولتوضيح تطبيق هذه الطريقة تم حل بعض المسائل كما تم التحقق من صحة الحلول الناتجة.

ABSTRACT

In the present paper, an exact analytic approach is obtained for solving inverse problems of steady, two-dimensional, heat conduction in a composite, plane or cylindrical media, by developing analytic methods of solving inverse problems for single-wall geometric shapes : plane wall and hollow cylinder. The present method is further extended to treat the problem if the composite media involves internal heat generation. Solved examples are presented to demonstrate application of the proposed method as well prove its validity.

1. INTRODUCTION

In the solution of direct heat conduction problems, the objective is to determinate the interior conditions (i.e., temperature and/or heat flux) when the boundary conditions are prescribed over the entire surface. Conversely, in the inverse problems one seeks the boundary conditions when the interior or the back surface conditions are prescribed [1]. Generally, direct problems occur mainly in design applications while inverse problems are encountered in analysis of experimental data. [1,2].

The inverse problems are classified in heat conduction literatures to steady, and transient problems [2]. In the last few years, there has been considerable interest in the solution of the transient inverse problems. Most of those studies have been performed numerically (e. g., [2-4]), while the analytic ones (e. g., [5,6]) are still scarce and restricted to the one-dimensional case due to the difficulty of a multi-dimensional solution [5]. Further, a limited number of approaches (e.g., [1]) are available in the literatures for the inverse solution of transient, one-dimensional problems in composite media.

Recently, exact analytic methods have been presented in a series of papers [7-11] to handle inverse problems in steady, two-dimensional heat conduction for simple shapes : a planar wall and a hollow cylindrical wall. The purpose of this paper is to develop an analytic approach for solving inverse problems of steady, two-dimensional heat conduction in a composite wall by extending analytic methods developed for a single plane wall [7,8], and for a single cylindrical wall [9,11].

2. MATHEMATICAL FORMULATION FOR A COMPOSITE PLANE WALL

Consider a two-dimensional, m-layered composite wall (height L and thickness X_m) consisting of m plane layers from different materials having constant thermal properties. Figure 1 shows the geometry and coordinate system. The distribution of each the temperature and its gradients at the exterior boundary surface (0,y); both are known continuous functions of the y coordinate. The main aim is to determine the (x,y) field of temperature in the composite wall using only these two known boundary conditions.

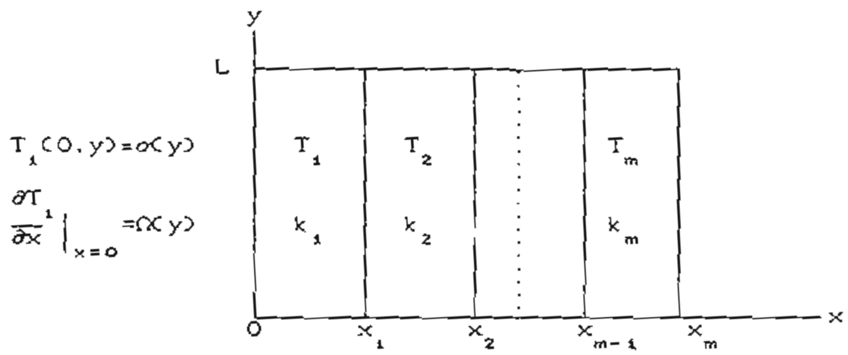

Fig. 1 The geometry and coordinate system for a composite plane wall

The mathematical formulation of this problem may be modeled by Laplace equation

$$\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} = 0 \quad \text{in } x_{i-1} \le x \le x_i \text{ and } 0 \le y \le L. \tag{1}$$

$$i = 1,2,3,\ldots,m \text{ and } x_o = 0.$$

with the known boundary conditions :

$$T_1(0,y) = \sigma(y), \tag{2a}$$

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=0} = -\frac{q_{x_1}(0,y)}{k_1} = \Omega(y) \tag{2b}$$

and the unknown interfacial boundary conditions :

$$T_i(x_i,y) = T_{i+1}(x_i,y), \quad \text{for } i = 1,2,3,\ldots,m-1 \tag{3a}$$

$$k_i \left.\frac{\partial T_i}{\partial x}\right|_{x_i} = k_{i+1} \left.\frac{\partial T_{i+1}}{\partial x}\right|_{x_i} \quad \text{with } i = 1,2,3,\ldots,m-1. \tag{3b}$$

Condition (3b) can be expressed in an alternative form as

$$q_{x_i}(x_i,y) = q_{x_{i+1}}(x_i,y), \quad \text{with } i = 1,2,3,\ldots,m-1 \tag{3b'}$$

where the notions : $\alpha(y)$ and $\beta(y)$ are known continuous functions of the y variable, k_i is the thermal conductivity of the i^{th} layer, and $q_{x_i}(x_i, y)$ is the x-direction heat flux in the i^{th} layer at the interfacial plane x_i.

2.1 Solution Procedure

On the light of an exact analytic method previously developed for solving a certain type of steady inverse problems in a single-layer, two-dimensional, plane wall [7], an analytic procedure is developed for solving such a stated inverse problem in a two-dimensional, composite wall (cf. Fig. 1). This procedure is performed in the following steps :

1- The (x,y) field of temperature of the first layer ($0 \le x \le x_1$ and $0 \le y \le L$) is calculated as in the case of single-wall [7] by

$$T_1(x,y) = \sum_{n=0}^{\infty} a_n(x) \frac{d^{2n}T_1(0,y)}{dy^{2n}} - \frac{1}{k_1}\sum_{n=0}^{\infty} b_n(x) \frac{d^{2n}q_{x_1}(0,y)}{dy^{2n}} \qquad (4a)$$

where

$$a_n(x) = \frac{(-1)^n x^{2n}}{(2n)!} \quad (4b), \qquad b_n(x) = \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (4c)$$

$T_1(0,y)$ and $q_{x_1}(0,y)$ are known boundary conditions defined by Eqs. (2a) and (2b), respectively.

2- Then, the x-direction heat flux field of the 1st layer is calculated by applying Fourier's law on Eq. (4a). This gives

$$q_{x_1}(x,y) = -k_1 \sum_{n=0}^{\infty} a'_n(x) \frac{d^{2n}T_1(0,y)}{dy^{2n}} + \sum_{n=0}^{\infty} b'_n(x) \frac{d^{2n}q_{x_1}(0,y)}{dy^{2n}} \qquad (5a)$$

where $a'_n(x)$ and $b'_n(x)$ are the first-order, x-derivatives of the $a_n(x)$ and $b_n(x)$ functions, respectively, which are defined by

$$a'_n(x) = \frac{(-1)^n x^{2n-1}}{(2n-1)!} \quad (5b) \quad \text{and} \quad b'_n(x) = \frac{(-1)^n x^{2n}}{(2n)!} \quad (5c)$$

3- The temperature $T_1(x_1,y)$ and heat flux $q_{x_1}(x_1,y)$ at the interfacial surface x_1 are calculated by Eqs. (4) and (5); with $x = x_1$, respectively.

4- According to the interfacial boundary conditions (3a) and (3b′), one sets $T_2(x_1,y) = T_1(x_1,y)$ and $q_{x_2}(x_1,y) = q_{x_1}(x_1,y)$

5- Now, the temperature solution of the 2nd layer ($x_1 \le x \le x_2$ and $0 \le y \le L$) is determined similar to that of the 1st layer by

$$T_2(x,y)=\sum_{n=0}^{\infty} a_n(x-x_1)\frac{d^{2n}T_2(x_1,y)}{dy^{2n}}-\frac{1}{k_2}\sum_{n=0}^{\infty} b_n(x-x_1)\frac{d^{2n}q_{x_2}(x_1,y)}{dy^{2n}} \quad (6a)$$

wherein

$$a_n(x-x_1)=\frac{(-1)^n(x-x_1)^{2n}}{(2n)!} \quad (6b),\qquad b_n(x-x_1)=\frac{(-1)^n(x-x_1)^{2n+1}}{(2n+1)!} \quad (6c)$$

where $q_{x_2}(x_1,y)$ and $T_2(x_1,y)$ are known from step 4.

6- Then, the x-direction heat flux field of the 2nd layer is determined by applying Fourier's law on Eq. (6). This yields

$$q_{x_2}(x,y)=-k_2\sum_{n=0}^{\infty} a_n'(x-x_1)\frac{d^{2n}T_2(x_1,y)}{dy^{2n}}+\sum_{n=0}^{\infty} b_n'(x-x_1)\frac{d^{2n}q_{x_2}(x_1,y)}{dy^{2n}} \quad (7a)$$

where

$$a_n'(x-x_1)=\frac{(-1)^n(x-x_1)^{2n-1}}{(2n-1)!} \quad (7b) \text{ and } b_n'(x-x_1)=\frac{(-1)^n(x-x_1)^{2n}}{(2n)!} \quad (7c)$$

7-In similar way, the procedure will be proceeded till the last layer of No. m.

However, the present solution can be generalized for any i^{th} layer of the m-layered composite media as follows :

I- The temperature field is defined by

$$T_i(x,y)=\sum_{n=0}^{\infty} a_n(x-x_{i-1})\frac{d^{2n}T_i(x_{i-1},y)}{dy^{2n}}-\frac{1}{k_i}\sum_{n=0}^{\infty} b_n(x-x_{i-1})\frac{d^{2n}q_{x_i}(x_{i-1},y)}{dy^{2n}} \quad (8a)$$

where

$$a_n(x-x_{i-1})=\frac{(-1)^n(x-x_{i-1})^{2n}}{(2n)!} \quad (8b) \text{ and } b_n(x-x_{i-1})=\frac{(-1)^n(x-x_{i-1})^{2n+1}}{(2n+1)!} \quad (8c)$$

II- The x-direction heat flux solution :

$$q_{x_i}(x,y)=-k_i\sum_{n=0}^{\infty} a_n'(x-x_{i-1})\frac{d^{2n}T_i(x_{i-1},y)}{dy^{2n}}+\sum_{n=0}^{\infty} b_n'(x-x_{i-1})\frac{d^{2n}q_{x_i}(x_{i-1},y)}{dy^{2n}} \quad (9)$$

for $i = 1,2,3,\dots,m$ in $x_{i-1} \le x \le x_i$, $x_o = 0$ and $0 \le y \le L$.

Here, it is important to point out that the solution procedure using the above expressions (8) and (9) have to be performed in subsequent steps starting with the first layer and ending up with the last layer number m.

2.2 Extending the Solution for Internal Heat Generation Case

If there is internal heat generation in the composite media, the heat conduction problem may be modeled by

$$\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} + \frac{\overset{*}{q}_i}{k_i} = 0, \quad \text{in } x_{i-1} \le x \le x_i \text{ and } 0 \le y \le L, \tag{11}$$

$$i = 1,2,3,\dots,m \text{ and } x_0 = 0.$$

with the same boundary conditions defined by Eqs. (2)-(3), where $\overset{*}{q}_i$ is the volumetric heat generation rate in the i^{th} layer.

On the light of the method of solving the corresponding inverse problem for a single plane wall [8], solution of problem (11) is obtained which can be expressed in the general form :

$$T_i(x,y) = T_i(x,y) \text{ [given by Eq. (8a)] } - \frac{\overset{*}{q}_i (x-x_{i-1})^2}{2k_i} \tag{12}$$

for $i = 1,2,3,\dots,m$ in $x_{i-1} \le x \le x_i$; $x_0 = 0$ and $0 \le y \le L$.

The solution procedure has to be performed in subsequent steps similar to that in section 2.1, as it will next be demonstrated by some application examples.

2.3 Application Examples

Example 1 : Consider a two-layered composite wall consisting of two different materials having constant thermal conductivity coefficients k_1 and k_2. The boundary surface (0,y) is thermally insulated and has temperature profile described by $T_1(0,y) = y^3+1$. Figure 2 illustrates the problem. It is required to determine the (x,y) solution of temperature and heat flux.

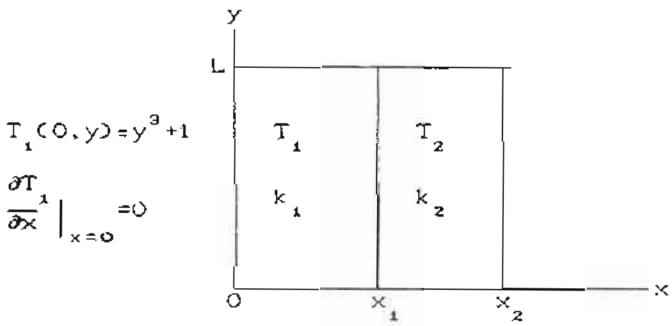

Fig. 2 Problem description of example 1

The mathematical model for the problem is

$$\frac{\partial^2 T_i}{\partial x^2} + \frac{\partial^2 T_i}{\partial y^2} = 0 \quad \text{in} \quad x_{i-1} \le x \le x_i \text{ and } 0 \le y \le L. \qquad (13)$$

$$i = 1,2 \text{ and } x_o = 0.$$

with

$$T_1(0,y) = y^3 + 1 \qquad (14a)$$

$$q_{x_1}(0,y) = 0 \qquad (14b)$$

$$T_2(x_1,y) = T_1(x_1,y) \quad \text{(unknown)} \qquad (14c)$$

$$k_2 \left.\frac{\partial T_2}{\partial x}\right|_{x_1} = k_1 \left.\frac{\partial T_1}{\partial x}\right|_{x_1} \text{ or } q_{x_2}(x_1,y) = q_{x_1}(x_1,y) \text{ (unknown)} \qquad (14d)$$

<u>The solution in layer No.1</u>

According to the boundary condition (14b), the general solution of temperature in the 1st layer, given by Eq.(4a), reduces to

$$T_1(x,y) = \sum_{n=0}^{\infty} a_n(x) \frac{d^{2n} T_1(0,y)}{dy^{2n}} \qquad (15a)$$

Substituting $a_n(x)$ and $T_1(0,y)$ from Eqs.(4b) and (14a), respectively, into the above equation gives

$$T_1(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n}}{dy^{2n}} \left\{ y^3 + 1 \right\} \qquad (15b)$$

Equation (15b) in expanded form is

$$T_1(x,y) = y^3 - 3x^2 y + 1 \qquad (16a)$$

Applying Fourier's law on the above equation gives the x-direction heat flux distribution as

$$q_{x_1}(x,y) = -k_1 \frac{\partial T_1(x,y)}{\partial x} = 6k_1 xy \qquad (16b)$$

and the y-direction heat flux field by

$$q_{y_1}(x,y) = -k_1 \frac{\partial T_1(x,y)}{\partial y} = 3k_1(x^2 - y^2) \qquad (16c)$$

The solution in layer No.2

The interfacial condition (14c) and Eq. (16a) give

$$T_2(x_1,y) = y^3 - 3x_1^2y + 1 \qquad (17)$$

Similarity, from Eqs. (14d) and (16b) one obtains

$$q_{x_2}(x_1,y) = 6k_1x_1y \qquad (18)$$

The general solution of temperature in the 2nd layer is described by Eqs. (6a-c). Hence, substituting $T_2(x_1,y)$, $q_{x_2}(x_1,y)$ from Eqs. (17) and (18), respectively, into Eqs. (6a-c) gives

$$T_2(x,y) = y^3 - 3x_1^2y - 3(x-x_1)^2y - 6k_1x_1(x-x_1)y/k_2 + 1 \qquad (19a)$$

Applying Fourier's law on Eq. (19a) yields

$$q_{x_2}(x,y) = 6k_2(x-x_1)y + 6k_1x_1y \qquad (19b)$$

and

$$q_{y_2}(x,y) = -k_2(3y^2 - 3x_1^2 - 3(x-x_1)^2) + 6k_1x_1(x-x_1) \qquad (19c)$$

Since the principle of heat balance are frequently used to test the validity as well estimate the accuracy of a numerical solution [12], we will apply this principle on the above solution to test the present method. This task is presented in Appendix (I). The result proves validity and exactness of the proposed method.

<u>Example 2</u> : Resolve example 1 assuming volumetric internal heat generation rate $\overset{*}{q}_1$ in the first layer and $\overset{*}{q}_2$ in second layer of the considered composite wall

This problem can be modeled by equation (11); for $i = 1,2$, under the boundary and interfacial conditions (14a)-(14d). Consequently, the general solution of temperature, given by Eq. (12), can be interpolated for each layer as follows :

The 1st layer solution

From combining Eq. (12) and Eq. (8); for i=1, with the boundary conditions (14b) one obtains

$$T_1(x,y) = \sum_{n=0}^{\infty} a_n(x) \frac{d^{2n}T_1(0,y)}{dy^{2n}} - \frac{\overset{*}{q}_1 x^2}{2k_1} \qquad (20a)$$

Inserting $a_n(x)$ and $T_1(0,y)$ from Eqs. (4b) and (14a), respectively, into Eq. (20a) yields

$$T_1(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \frac{d^{2n}}{dy^{2n}} \left\{y^3 + 1\right\} - \frac{\dot{q}_1 x^2}{2k_1} \quad (20b)$$

The above equation in expanded form yields

$$T_1(x,y) = y^3 - 3x^2y + 1 - \frac{\dot{q}_1 x^2}{2k_1} \quad (21a)$$

Consequently, the heat flux components can be calculated from the above equation as :

$$q_{x_1}(x,y) = -k_1 \frac{\partial T_1(x,y)}{\partial x} = 6k_1xy + \dot{q}_1 x \quad (21b)$$

and

$$q_{y_1}(x,y) = -k_1 \frac{\partial T_1(x,y)}{\partial y} = 3k_1(x^2 - y^2) \quad (21c)$$

The 2nd layer solution

The interfacial condition (14c) with Eq. (21a) yields

$$T_2(x_1,y) = y^3 - 3x_1^2y + 1 - \frac{\dot{q}_1 x_1^2}{2k_1} \quad (22a)$$

Also the interfacial condition (14d) with Eq. (21b) gives

$$q_{x_2}(x_1,y) = 6k_1x_1y + \dot{q}_1 x_1 \quad (22b)$$

Combining Eqs. (12) and (8); with $i=2$, and substituting Eqs. (22a) and (22b) yields

$$T_2(x,y) = y^3 - 3y[x_1^2 + (x-x_1)^2 + 2k_1x_1(x-x_1)/k_2] + 1 - \frac{\dot{q}_1 x_1^2}{2k_1} - \frac{\dot{q}_1 x_1(x-x_1)}{k_2} - \frac{\dot{q}_2 (x-x_1)^2}{2k_2} \quad (23a)$$

Applying Fourier's law on the above equation gives

$$q_{x_2}(x,y) = 6(k_2(x-x_1) + k_1x_1)y + \dot{q}_1 x_1 + \dot{q}_2 (x-x_1) \quad (23b)$$

and

$$q_{y_2}(x,y) = 3k_2(x_1^2 - y^2 + (x-x_1)^2) + 6k_1x_1(x-x_1) \quad (23c)$$

Exactness of the above solution has also been examined by calculating the heat balance of the system. The test result indicates validity and exactness of the proposed approach.

3. MATHEMATICAL FORMULATION FOR A COMPOSITE CYLINDRICAL WALL

Figure 3 describes the studied problem for a composite cylindrical wall consisting of m layers from different materials having constant thermal and electrical properties. This type of the inverse problems is characterized by existing two known boundary conditions at one boundary surface : the axial distribution of each the temperature and the exterior heat flux; both are prescribed at the inner cylinder surface by known continuous and differentiable functions of the axial coordinate y.

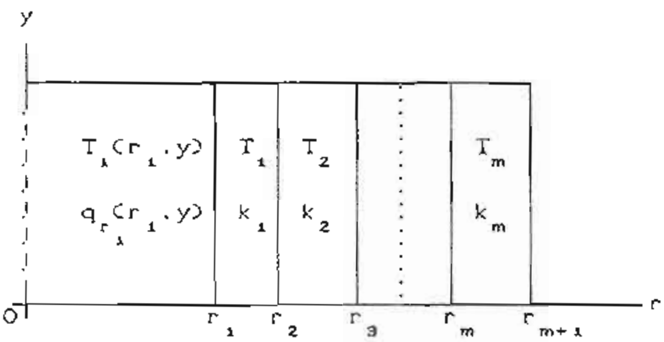

Fig. 3 The stated inverse problem for a cylindrical composite wall

The mathematical formulation of this problem is described by

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r} + \frac{\partial^2 T_i}{\partial y^2} = 0 \quad \text{in} \quad r_i \le r \le r_{i+1}, \quad 0 \le y \le L \qquad (24a)$$

$$i = 1,2,3,\ldots,m$$

with

$$T_1(r_1,y) = \zeta(y) \qquad (24b)$$

$$q_{r_1}(r_1,y) = \beta(y) \qquad (24c)$$

(known)

$$q_{r_i}(r_{i+1},y) = q_{r_{i+1}}(r_{i+1},y), \qquad (24d)$$

$$T_i(r_{i+1},y) = T_{i+1}(r_{i+1},y) \qquad (24e)$$

(unknown) $i = 1,2,3,\ldots,m-1$

where the notions : $\zeta(y)$ and $\beta(y)$ are known continuous functions of the y variable, and k_i is the thermal conductivity of the i^{th} layer. $T_i(r_{i+1},y)$ and $q_{r_i}(r_{i+1},y)$ are the temperature and the radial heat flux, respectively, in the i^{th} layer at the interfacial surface r_{i+1}.

Similar to the previously described case of the composite plane wall, the solution of the above problem can be found by extending the known solution for a single cylindrical wall [9]. Thus, the solution of this problem can be obtained which is expressed by:

$$T_i(r,y) = \sum_{n=0}^{\infty} F_n[r]_i \frac{d^{2n}T_i(r_i,y)}{dy^{2n}} - \frac{1}{k_i}\sum_{n=0}^{\infty} G_n[r]_i \frac{d^{2n}q_{r_i}(r_i,y)}{dy^{2n}} \quad (25a)$$

where the leading terms of $F_n[r]_i$ are :

$$\left.\begin{aligned}
F_0[r]_i &= 1, \quad F_1[r]_i = \frac{r_i^2}{4}\left[1 - \left(\frac{r}{r_i}\right)^2 + 2\ln\left(\frac{r}{r_i}\right)\right] \\
F_2[r]_i &= -\frac{r_i^4}{16}\left[\frac{5}{4} - \left(\frac{r}{r_i}\right)^2 - \frac{1}{4}\left(\frac{r}{r_i}\right)^4 + \left(1+2\left(\frac{r}{r_i}\right)^2\right)\ln\left(\frac{r}{r_i}\right)\right] \\
F_3[r]_i &= \frac{r_i^6}{64}\left[\frac{10}{36} + \frac{1}{4}\left(\frac{r}{r_i}\right)^2 - \frac{1}{2}\left(\frac{r}{r_i}\right)^4 - \frac{1}{36}\left(\frac{r}{r_i}\right)^6 + \right. \\
&\qquad \left.\left(\frac{1}{6} + \left(\frac{r}{r_i}\right)^2 + \frac{1}{2}\left(\frac{r}{r_i}\right)^4\right)\ln\left(\frac{r}{r_i}\right)\right]
\end{aligned}\right\} \quad (25b)$$

and that of $G_n[r]_i$ are :

$$\left.\begin{aligned}
G_0[r]_i &= r_i\ln\left(\frac{r}{r_i}\right), \\
G_1[r]_i &= -\frac{r_i^3}{4}\left[1 - \left(\frac{r}{r_i}\right)^2 + \left(1+\left(\frac{r}{r_i}\right)^2\right)\ln\left(\frac{r}{r_i}\right)\right] \\
G_2[r]_i &= +\frac{r_i^5}{64}\left[\frac{3}{2} - \frac{3}{2}\left(\frac{r}{r_i}\right)^4 + \left(1+4\left(\frac{r}{r_i}\right)^2+\left(\frac{r}{r_i}\right)^4\right)\ln\left(\frac{r}{r_i}\right)\right] \\
G_3[r]_i &= -\frac{r_i^7}{128}\left[\frac{11}{108} + \frac{1}{4}\left(\frac{r}{r_i}\right)^2 - \frac{1}{4}\left(\frac{r}{r_i}\right)^4 - \frac{11}{108}\left(\frac{r}{r_i}\right)^6 + \right. \\
&\qquad \left.\left(\frac{1}{18} + \frac{1}{2}\left(\frac{r}{r_i}\right)^2 + \frac{1}{2}\left(\frac{r}{r_i}\right)^4 + \frac{1}{18}\left(\frac{r}{r_i}\right)^6\right)\ln\left(\frac{r}{r_i}\right)\right]
\end{aligned}\right\} \quad (25c)$$

for $i = 1,2,3,\ldots,m$ in $r_i \le r \le r_{i+1}$ and $0 \le y \le L$.

3.1 Extending the Solution for Internal Heat Generation Case

If the composite cylinder wall involves heat generation , the above solution can be extended to treat the modified problem by following analysis way similar to that made in the case of a composite plane wall. Thus, the solution becomes

$$T_i(r,y) = T_i(r,y)\ [\text{by Eq. (25a)}] - \frac{q^*_i r_i^2}{4k_i}\left[1 - \left(\frac{r}{r_i}\right)^2 + 2\ln\left(\frac{r}{r_i}\right)\right] \quad (26)$$

for $i = 1,2,3,\ldots,m$ in $r_i \le r \le r_{i+1}$ and $0 \le y \le L$.

It is noted that Eq. (26) with $i = 1$, yields the same known solution found for a single cylindrical wall involving internal heat generation [11].

Here. it is important to remember that the solution has to be performed in marching procedure starting with the first layer and ending up with the last layer of No. m; similar to that followed in the case of the composite-plane=wall solution.

Notice : if the two known boundary conditions (24b) and (24c); required for the method application. are prescribed at the outside surface of the hollow cylinder instead at the inner surface as in Fig. 3. the solution will be different [11]. However. the above expressions (25)-(26) can also be used as the solution of this new case provided reversing the consequence of numbering the layers and radii in Fig. 3; i.e., the layer of No. 1 is the outer layer of radius r_1. and the last layer of No. m is the inner layer of radius r_m.

3.2 Application Example

Consider a hollow cylinder with a composite wall from two different materials having constant thermal properties. Figure 4 states the problem. The inner cylinder surface is thermally insulated and has temperature varying with the axial location. which described by $T_1(r_1,y) = ay^3+b$. where a and b are known constants.

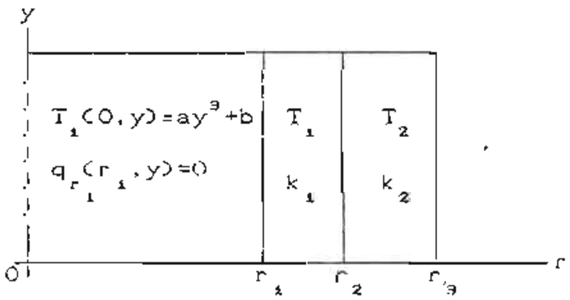

Fig. 4 The problem statement of the example

The mathematical formulation of this problem is described by

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{1}{r}\frac{\partial T_i}{\partial r} + \frac{\partial^2 T_i}{\partial y^2} = 0 \quad \text{in} \quad r_i \le r \le r_{i+1}, \quad 0 \le y \le L \qquad (27a)$$

$$i = 1,2$$

with

$$T_1(r_1,y) = ay^3 + b \qquad (27b)$$

$$q_{r_1}(r_1,y) = 0 \qquad (27c)$$

$$T_2(r_2,y) = T_1(r_2,y) \qquad (27d)$$

$$q_{r_2}(r_2,y) = q_{r_1}(r_2,y) \qquad (27e)$$

} (unknown)

The solution in the 1st layer

Equation (25a) with i =1, gives the general solution of temperature in the 1st layer, which under the boundary conditions (27b) and (27c) yields

$$T_1(r,y) = \sum_{n=0}^{\infty} F_n[r]_i \frac{d^{2n}}{dy^{2n}} \left[ay^3 + b\right] \tag{28}$$

Equation (28) in expanded form yields

$$T_1(r,y) = (ay^3+b)\ F_0[r]_1 + 6ay\ F_1[r]_1 \tag{29a}$$

From Eq. (25b) with i = 1, the F[r] functions in Eq. (29a) are

$$F_0[r]_1 = 1, \qquad F_1[r]_1 = \frac{r_1^2}{4}\left[1 - (\frac{r}{r_1})^2 + 2\ln(\frac{r}{r_1})\right] \tag{29b}$$

Consequently, the radial heat flux is calculated from Eq. (29a) as

$$q_{r_1}(r,y) = -k_1 \frac{\partial T_1}{\partial r} = -6ak_1 yF'_1[r]_1 \tag{29c}$$

wherein $F'_1[r]_1$ is the 1st-order, radial derivative of $F_1[r]_1$ which is calculated from Eq. (29b), by

$$F'_1[r]_1 = \frac{r_1}{2}\left[\frac{r_1}{r} - \frac{r}{r_1}\right] \tag{29d}$$

By similarity, the y-direction heat flux calculated from Eq. (29a) is

$$q_{y_1}(r,y) = -3ak_1\ (y^2 + 2F_1[r]_1) \tag{29e}$$

The solution in the 2nd layer

According to Eqs. (27d) and Eq. (29a) one obtains

$$T_2(r_2,y) = (ay^3+b) + 6ayF_1[r_2]_1 \tag{30}$$

By similarity, Eqs. (27e) and (29c) gives

$$q_{r_2}(r_2,y) = -6ak_1 y\ F'_1[r_2]_1 \tag{31}$$

where $F_1[r_2]_1$ and $F'_1[r_2]_1$ are calculated from Eqs. (29b) and (29d), respectively; with setting $r = r_2$.

Now, by setting i = 2 in Eq. (25a), the general solution of temperature reads

$$T_2(r,y) = \sum_{n=0}^{\infty} F_n[r]_2 \frac{d^{2n}T_2(r_2,y)}{dy^{2n}} - \frac{1}{k_2}\sum_{n=0}^{\infty} G_n[r]_2 \frac{d^{2n}q_{r_2}(r_2,y)}{dy^{2n}} \quad (32a)$$

By substituting $T_2(r_2,y)$ and $q_{r_2}(r_2,y)$ from Eqs. (30) and (31), respectively, into the above equation, one gets

$$T_2(r,y) = (ay^3+b) + 6ay\left\{F_1[r_2]_1 + F_1[r]_2 + k_1/k_2 G_0[r]_2 \; F_1'[r_2]_1\right\} \quad (32b)$$

where $F_1[r]_2$ reads from Eq. (25b); with $i = 2$, as

$$F_1[r]_2 = \frac{r_2^2}{4}\left[1 - \left(\frac{r}{r_2}\right)^2 - 2\ln\left(\frac{r}{r_2}\right)\right] \quad (32c)$$

and $G_0[r]_2$ from Eq. (25c); with $i = 2$, by

$$G_0[r]_2 = r_2 \ln(r_2/r) \quad (32d)$$

Applying Fourier's law on Eq. (32b) gives the radial heat flux as

$$q_{r_2}(r,y) = -6ay\left\{k_2 F_1'[r]_2 + k_1 G_0'[r]_2 \; F_1'[r_2]_1\right\} \quad (33a)$$

where $F_1'[r]_2$ is calculated from Eq. (32c) as

$$F_1'[r]_2 = \frac{r_2^2}{2}\left[\frac{r_2}{r} - \frac{r}{r_2}\right] \quad (33b)$$

and $G_0'[r]_2$ from Eq. (32d) by

$$G_0'[r]_2 = (r_2/r) \quad (33c)$$

By similarly, one gets the y-direction heat flux from Eq. (32b) by

$$q_{y_2}(r,y) = -3ak_2y^2 - 6a\left\{k_2 F_1[r_2]_1 + k_2 F_1[r]_2 + k_1 G_0[r]_2 \; F_1'[r_2]_1\right\} \quad (34)$$

Check on the above solution is presented in Appendix (II).

4 CONCLUSIONS

The present study provides an exact and direct method for solving inverse problems of steady, two-dimensional, heat conduction in a composite media from the knowledge of the temperature, and exterior heat flux distribution,provided that both are prescribed at the same boundary surface by known continuous and differentiable functions of the spatial variable with the surface length. The effect of heat generation in the wall is exactly considered in the solution. Explicit expressions for temperature, and heat flux calculation are obtained for a plane composite wall as well for a cylindrical composite wall.

APPENDIX (I)

Here, solution of example No. 1 which is obtained by the present method, will be checked up by applying the heat balance on the wall.

The surface heat flux components, shown in Fig. 5, can be determined as follows :

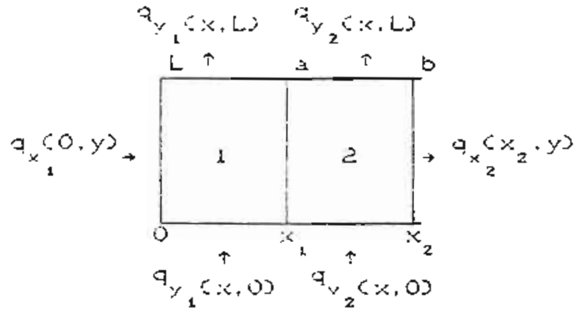

Fig. 5 Heat balance of App. I

$$q_{x_1}(0,y) = 0; \qquad 0 \le y \le L \tag{35}$$

$$q_{y_1}(x,0) = 3k_1x^2; \qquad 0 \le x \le x_1 \tag{36}$$

$$q_{y_2}(x,0) = 3k_2(x_1^2+(x-x_1)^2)+6k_1x_1(x-x_1); \qquad x_1 \le x \le x_2 \tag{37}$$

$$q_{x_2}(x_2,y) = 6k_2(x_2-x_1)y + 6k_1x_1y; \qquad 0 \le y \le L \tag{38}$$

$$q_{y_1}(x,L) = 3k_1(x^2-L^2); \qquad 0 \le x_1 \le x \tag{39}$$

$$q_{y_2}(x,L) = k_2(-3L^2+3x_1^2+3(x-x_1)^2)+6k_1x_1(x-x_1); \qquad x_1 \le x \le x_2 \tag{40}$$

by using Eqs. (18b) to (19c), respectively.

The heat flow into the system of Fig.5 is calculated by

$$\text{Heat inflow} = \int_0^L q_{x_1}(0,y)\,dy + \int_0^{x_1} q_{y_1}(x,0)\,dx + \int_{x_1}^{x_2} q_{y_2}(x,0)\,dx \tag{41}$$

By substituting Eqs. (35) to (37) into Eq. (41) with calculating the integrals, one gets

$$\text{Heat inflow} = k_1(x_1^3+ 3x_1(x_2-x_1)^2)+k_2((3x_1^2x_2- 3x_1^3+ (x_2-x_1)^3) \tag{42}$$

The heat flow out from the wall is calculated by :

$$\text{Heat outflow} = \int_0^L q_{x_2}(x_2,y)\,dy + \int_0^{x_1} q_{y_1}(x,L)\,dx + \int_{x_1}^{x_2} q_{y_2}(x,L)\,dx \tag{43}$$

Substituting Eqs.(38) to(40) into Eq.(43),with calculating the involved integrals,one obtains the same result as that of Eq. (42). This means that inflow is equal to heat outflow. This proves exactness and validity of the proposed method.

APPENDIX (II)

In this context, the solution of the application example, derived in section 3.2, will be checked by calculating the heat balance of the system illustrated in Fig. 6

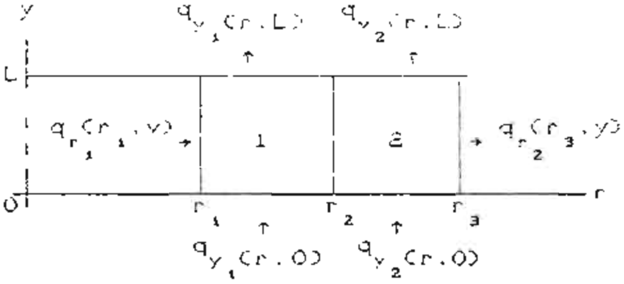

Fig. 6 Heat balance of App. II

The net heat flow in the x-direction is calculated by

$$\sum_{out} Q_x - \sum_{in} Q_x = 2\pi \int_0^L \left\{ r_3\, q_{r_2}(r_3,y) - r_1\, q_{r_1}(r_1,y) \right\} dy \qquad (44)$$

Substituting $q_{r_1}(r_1,y)$ and $q_{r_2}(r_3,y)$ from Eq. (29c); with $r=r_1$, and Eq. (33a); with $r = r_3$, respectively, into Eq. (44) yields

$$\sum_{out} Q_x - \sum_{in} Q_x = 6a\pi r_3 L^2 \left[k_2\, F_1'[r_3]_2 + k_1\, G_0'[r_3]_2\, F_1'[r_2]_1 \right] \qquad (45)$$

Substituting $F_1'[r_3]_2$, $G_0'[r_3]_2$ and $F_1'[r_2]_1$ from Eqs. (33b), (33c) and (29d) respectively, into Eq. (45) gives

$$\sum_{out} Q_x - \sum_{in} Q_x = 3a\pi L^2 \left[k_2 (r_3^2 - r_2^2) + k_1 (r_2^2 - r_1^2) \right] \qquad (46)$$

In similar way, the net heat flow in y-direction is

$$\sum_{out} Q_y - \sum_{in} Q_y = 2\pi \int_{r_1}^{r_2} \left\{ q_{y_1}(r,L) - q_{y_1}(r,0) \right\} r\,dr + 2\pi \int_{r_2}^{r_3} \left\{ q_{y_2}(r,L) - q_{y_2}(r,0) \right\} r\,dr \qquad (47)$$

Substituting from Eqs. (29e) and (34) into Eq. (47) gives

$$\sum_{out} Q_y - \sum_{in} Q_y = -3a\pi L^2 \left[k_1 (r_2^2 - r_1^2) + k_2 (r_3^2 - r_2^2) \right] \qquad (48)$$

Equations (46) and (48) indicate that heat inflow is equal to heat outflow. This proves validity of the proposed method.

NOMENCLATURES

$a_n(x)$	x-dependent coefficient, m^{2n}
$b_n(x)$	x-dependent coefficient, m^{2n+1}
$F_n[r]$	r-dependent coefficient, m^{2n}
$G_n[r]$	r-dependent coefficient, m^{2n+1}

k_i	thermal conductivity of the layer No. i, kW/(m °C)
L	wall height, m
m	No. of layers in the composite media, dimensionless
q_{r_i}	radial heat flux in the layer No. i, kW/m^2
q_{x_i}	x-direction heat flux in the layer No. i, kW/m
q_{y_i}	y-direction heat flux in the layer No. i, kW/m^2
$\dot{q}$	volumetric heat generation rate, kW/m^3
r	radial coordinate, m
T_i	temperature in the layer No. i, °C
x, y	cartesian coordinates, m
$\alpha(y)$, $\zeta(y)$	y-dependent functions, °C
$\Omega(y)$	y-dependent function, °C/m

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