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GENERALISED SIMPLE EIGENVALUES AND BIFURCATION FOR A LINKED MULTIPARAMTER EIGENVALUE PROBLEM

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1. Introduction

Suppose that we are given a multiparameter system of equations,

$$L_i(\underline{\lambda})x_i = f(\underline{\lambda}, x_1, \ldots, x_m) \quad ; \qquad (1.1)$$

$$L_i(\underline{\lambda}) = A_i - \sum_{j=1}^{n} \lambda_j B_{ij}, \qquad i = 1, \ldots, m \qquad m \le n$$

where A_i, B_{ij} are bounded self-adjoint operators on Hilbert spaces H_i, $i = 1, \ldots, m$ and λ_j, $j = 1, \ldots, n$ are real paramters. An eigenvalue of (1.1) is a point $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ for which each equation possess a solution $x_i \neq 0$. The spectral theory of multiparameter system of the case m=n has been considered in many recent papers, see e.g. [1,6,8] for abstract problem and in [2,3,4,5] for a linked system of non-liner second order ordinary differential equation. In this paper we are concerned with the problem of bifurcation of solutions of the non-linear problem (1.1) at a generalised simple eigenvalue of the linearised problem for the case $m \le n$. The result is obtained using the concept of generalised simple eigenvalues that we have introduced in [7]. Using this concept we also invstigate the multiparameter eigenvalue problem in the case where some of the operators depend analytically on a perturbation parameter. The distribution of this paper is as follows:

In section 2 we give our definition for the generalised simple eigenvalue. In section 3. Theorem 3.2, show that, under some standard conditions on the nonlinear terms, (1.1) has a set of solutions bifurcating from the trivial solution $(\lambda,0) \in \Re^n \times X$ at such generalised simple eigenvalues. Theorem 3.3, gives the same results when some of the operators depend analytically on perturbation parameter.

2. Definition of a generalised simple eigenvalue

Let X, Y be real banach spaces and let a, B_i, i = 1, . . . ,n be bounded linear operators from X into Y. Consider the following problem

$$L(\underline{\lambda})\ x + N(\underline{\lambda}, x) = 0 \qquad (2.1)$$

where

$$L(\underline{\lambda}) = A - \sum_{j=1}^{n} \lambda_j B_j \qquad (2.2)$$

and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Re^n$

In [7] we have introduced the following definition of a generalised simple eigenvalue.

Definition 2.1

$\underline{\lambda}^\circ = (\lambda_1^\circ, \ldots, \lambda_n^\circ) \in \Re^n$ is a gteneralised simple eigenvalue for $(A, B_1, \ldots, B_n)$ if;

(i) dim $N(L(\underline{\lambda}^\circ)) = 1$;
(ii) $L(\underline{\lambda}^\circ)$ is a fredholm operator of index 1-m where $m \le n$;
(iii) $B_i x_\circ \notin \Re(L(\underline{\lambda}^\circ))$, $i = 1, \ldots, n$ where
$x_\circ \in N(L(\underline{\lambda}^\circ))$
and $Y = \text{span}\{B_i x_\circ,\ i = 1, \ldots, n\} \oplus \Re(L(\lambda^\circ))$.

By using definition 2.1 we have proved in [7] the following result.

THEOREM 2.2

Let $\underline{\lambda}^{\circ} \in \Re^{n}$ alised simple eigenvalue of (2.1) and let $N:\Re^{n} \times X \to Y$ be a non-linear mapping such that :

$$C1: N \in C^{r}(\Re^{n} x \quad X, Y) \qquad , r \geq 2;$$

$$C2: N(\underline{\lambda}, 0) = 0 \ ;$$

$$C3: D_{x}N(\underline{\lambda}_{m}, \underline{\mu}^{\circ}_{m}), 0 \qquad = 0$$

where

$$\underline{\lambda}_{m} = (\lambda_{1}, \ldots, \lambda_{m}), \qquad \underline{\mu}_{m} = (\underline{\lambda}_{m+1}, \ldots, \underline{\lambda}_{n}).$$

Then $(\underline{\lambda}^{\circ}, 0) \in \Re^{n} x \quad X$ is a bifurcation point of solutions of (2.1) and there exists a set of solutions

$$\{(\underline{\lambda}, x) = ((\underline{\lambda}^{*}_{m}(u, \mu_{m}), \mu_{m}), x^{*}(u, \mu_{m}));$$
$$u \in (-\delta, \delta) \subset \Re \qquad \text{for some } \delta > 0;$$
$$\left\| \underline{\mu}_{m} - \underline{\mu}^{\circ}_{m} \right\| < \varepsilon \text{ for some } \varepsilon > 0\}$$

where

$$\underline{\lambda}^{*}_{m} : \Re \times \Re^{n-m} \to \quad \Re^{m} \qquad \text{and}$$
$$x^{*} : \Re \times \Re^{n-m} \to \quad X$$

are C^{r-1} mapping.

We shall use the results of THEOREM 2.2 to study the linked system (1.1).

3. Linked Multiparameter Eigenvalue Proplems

Consider the system

$$M_{i}(\underline{\lambda}, x): \ = \ (A_{i} - \sum_{j=1}^{n} \lambda_{j} B_{ij}) x_{i} - f_{i}(\underline{\lambda}, x) \ = \ 0;$$

$i = 1,...,m; \quad m \leq n$ where $x = (x_1,...,x_m), \quad x_i \in H_i, A_i, B_{ij}$

are all Hermitian (bounded and self-adjoint linear operators) and

$f_i : \Re^n x H_i \rightarrow H_i$ is a non-linear mapping.

Define $X = \overset{m}{\underset{i=1}{x}} H_i$ which is an inner product space when given the inner product.

$$(x,y) = \sum_{i=1}^{m} (x_i, y_i)_{H_i} \qquad , x_i, y_i \in H_i.$$

It is clear that x with this inner product becomes a Hilbert space.
Follow binding [1] (see also [7]) we define the following operators:

$$A : X \rightarrow X \quad ; \qquad B_j : X \rightarrow X, \quad J = 1,...,n$$

where

$$Ax = (A_1 x_1,...,A_m x_m);$$
$$B_j x = (B_{1j} x_1,...,B_{mj} x_m);$$

and

$$f(\underline{\lambda}, x) = (f_1(\underline{\lambda}, x),...,f_m(\underline{\lambda}, x))$$

The system (3.1) is then equivalent to the single problem

$$M(\underline{\lambda}, x) := w(\underline{\lambda})x - f(\underline{\lambda}, x) = 0$$

where

$$W(\underline{\lambda}) := A - \sum_{i=1}^{n} \lambda_j B_j$$

With this notations, we obtian the following result :

LEMMA 3.1

Assume that for $\underline{\lambda} = \underline{\lambda}^{\circ} \in \Re^n$ each of the linear problem in the system (3.1) has exactly one llinear independent solution

X_i°, $i = 1,....,m$;$\Re(L_i(\underline{\lambda}))$ is closed and the matrix
$S(x^\circ) = S(x_1^\circ,...,x_m^\circ) = ((B_{ij}x_i^\circ, x_i^\circ)), i = 1,...,m; j = 1,...,n$

is such that all the determinants of order m not equal zero. Then the equivalent equation

$$w(\underline{\lambda})x = (A - \sum_{j=1}^{n} \lambda_j B_j)x = 0$$

satisfy the following :

(i) $w(\underline{\lambda})$is a sel-adjoint operator for all $\underline{\lambda} \in \Re^n$

(ii) $w(\underline{\lambda}^\circ)$is a fredholm operator with zero index;

$\dim N(L(\underline{\lambda}^\circ)) = m$ where
$N(L(\underline{\lambda}^\circ)) = spen[\zeta_1^\circ,...,\zeta_m^\circ], \zeta_i^\circ = (0,..0, x_i^\circ \quad ,0..0)$

and

$Y_1 = R(L(\underline{\lambda}^\circ)) = [spen[\zeta_1^\circ,...,\zeta_m^\circ]]^\perp$

(iii) $B_j x^\circ \notin Y_1; x^\circ = (x_1^\circ,...,x_m^\circ), j = 1,...,n$

and

$\dim \quad Y_\circ = m \leq n$ where
$Y_\circ = spen \quad [B_1x^\circ,...,B_nx^\circ]$

Hence $X = Y_\circ \oplus Y_1$

PROOF:

(i)and (ii) are easy and we shall prove (iii). since all the determinant of order m not equal zero ,then for each j there exist

$$(B_{ij}x_i^\circ, x_i^\circ) \neq 0 \rightarrow (B_j x^\circ, \zeta_i^\circ) \neq 0 \rightarrow B_j x^\circ \notin Y_1$$

Also,since rank $S(x^\circ) = m \leq n$, then dim $Y_\circ = m$ dim and hence :

$$X = Y_\circ \oplus Y_1$$

This lemma is exactly the result of [8] for the case m=n
We also note that x has the following direct sum

$$X = X_\circ \oplus X_1$$

where

$$X_\circ = \text{spec}(x^\circ)$$

and $$X_1 = \left[\text{spen}[\zeta_1,\ldots,\zeta_m^\circ]\right]^\perp$$

For the non-linear equation (3.2) we have the following:

THEOREM 3.2

Let $\underline{\lambda}^\circ \in \Re^n$ be as in lemma 3.1 and let $f_i ; \Re^n x \;\; X \rightarrow H_i , i = 1,\ldots,m$

satisfy;

C1 $f_i \in C^r(\Re^n x \;\; X, H_i), r \geq 2$

C2 $f_i(\underline{\lambda},0) = 0 \quad \forall \quad \underline{\lambda} \in \Re^n$

C3 $D_x f_i((\underline{\lambda}_m, \underline{\mu}_m^\circ)0) = 0$

then $(\underline{\lambda}^\circ, 0) \in \Re^n x \;\; X$ is a bifurcation point for solutions of (3.2) and there exists a set of solutions

$$\{(\underline{\lambda}, x) = (\underline{\lambda}_m^*(u,\underline{\mu}_m),\underline{\mu}_m); x_1^*(u,\underline{\mu}_m),\ldots,x_m^*(u,\underline{\mu}_m): \\ u \in (-\delta,\delta) \subset \Re \quad \text{for some} \quad \delta > 0\}$$

where

$\underline{\lambda}_m^* ; \Re x \Re^{n-m} \rightarrow \Re^m$

and $x_i^* ; \Re x \Re^{n-m} \rightarrow H_i , i = 1,\ldots,m$

are C^{r-1} mappings.

Proof:

The results follow by an application of the method of liapunou-schmidt ,as in [7] to fines solution of the form:

$x = u \; x^\circ + x^1$ where $u \in \Re$ $x^1 \in X_1$

For the sake of completeness we will proceed with the proof, suitably modified for our problem. let Q_o and Q_1 and be the projections of Y onto Y_o and Y_1 respectively (see(3.4)). Then

$$\begin{aligned} & M(\underline{\lambda};x)=0 \\ & \Leftrightarrow Q_1 M(\underline{\lambda};x)=0 \text{ and } Q_o M(\underline{\lambda};x)=0 \end{aligned} \tag{3.6}$$

the so called auxiliary equation and bifurcation equation respectively. The auxiliary equation becomes

$$Q_1 w(\underline{\lambda})x_1 - Q_1 f((\underline{\lambda}_m,\underline{\mu}_m),ux^\circ + x^1)=0 \tag{3.7}$$

where

$$u \in \Re, x^1 \in X_1.$$

consider the mapping $\psi : \Re \times \Re^{n-m} \times \Re \times X_1 \to Y_1$ defined by

$$\psi(\underline{\lambda}_m,\underline{\mu}_m u, x^1) = Q_1 w(\underline{\lambda})x_1 - Q_1 f((\underline{\lambda}_m,\underline{\mu}_m),ux^\circ + x^1)=0$$

Using C2 and C3 we obtain

$$\begin{aligned} & \psi(\underline{\lambda}_m^\circ,\underline{\mu}_m^\circ,0,0)=0, \\ & D_{x^1}\ \psi\ (\underline{\lambda}_m^\circ,\underline{\mu}_m^\circ,0,0)=Q_1 w(\underline{\lambda}^\circ). \end{aligned}$$

since $Q_1 W(\underline{\lambda}^\circ) : X_1 \to y_1$ is a bounded linear isomorphism, it follows from the implicit function theorem that there exists a neighborhood

$U \subset \Re^m \times \Re^{n-m} \times \Re$ of $(\underline{\lambda}_m^\circ,\underline{\mu}_m^\circ,0)$ and a unique mapping $x^{1*} \in C(U,X_1)$ such that

$$x^{1*}(\underline{\lambda}_m^\circ,\underline{\mu}_m^\circ,0)=0$$

and

$$\psi(\underline{\lambda}_m,\underline{\mu}_m,u,x^{1*}(\underline{\lambda}_m,\underline{\mu}_m,u))=0$$

i.e the auxiliary equation (3.8) is satisfied .since ,by C2 $(\underline{\lambda}_m^\circ,\underline{\mu}_m^\circ,0,0)$

satisfies (3.7) and ,by the implicit function theorem $\mathbf{x}^{1*}$ is unique ,it follows that

$$x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, 0) = 0 \ \ \forall \ \ (\underline{\lambda}_m, \underline{\mu}_m, 0) \in U.$$

Differentiation of (3.9) with respect to u and using C3 and (3.10) gives

$$Q_1 \, W(\underline{\lambda}_m, \underline{\mu}^{\circ}_m) D_u x^{1*}(\underline{\lambda}_m, \underline{\mu}^{\circ}_m, 0) = 0.$$

Since for $\|\underline{\lambda} - \lambda^{\circ}\|$ sufficiently small $Q_1 \, W(\underline{\lambda}_m, \underline{\mu}^{\circ}_m)$ is a bounded linear isomorphism of X_1 onto Y_1 we can conclude that

$$D_u x^{1*}(\underline{\lambda}_m, \underline{\mu}^{\circ}_m, 0) = 0 \qquad \text{for} \qquad \|\underline{\lambda}_m - \underline{\lambda}^{\circ}_m\| \text{ sufficiently small}$$

Differentiating (3.10) repeatedly with respect to $\underline{\mu}_m$ gives

$$D^k_{\mu_m} x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, 0) = 0, \qquad 1 \le k \le r.$$

Using (3.10) - (3.12) we see, from Taylors theorem

$$x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, u) = 0(|u|^2 + |u| \, \|\underline{\mu}_m - \underline{\mu}^{\circ}_m\|) \ \text{ as } \ |u| \ , \ \|\underline{\mu}_m - \underline{\mu}^{\circ}_m\| \to 0.$$

where $\quad x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, u) = (x^{1*}_1(\underline{\lambda}_m, \underline{\mu}_m, u), \ldots, x^{1*}_m(\underline{\lambda}_m, \underline{\mu}_m, u)).$

The bifurcation equation becomes

$$Q_{\circ} W(\underline{\lambda})(ux^{\circ} + x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, u)) - Q_{\circ} f(\underline{\lambda}_m, \underline{\mu}_m, ux^{\circ} + x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, u)) = 0$$

$$\Rightarrow -u\sum_{j=1}^{m}(\lambda_j - \lambda^{\circ}_j)B_j x^{\circ} - u \sum_{j=m+1}^{n}(\lambda_j - \lambda^{\circ}_j)B_j x^{\circ} - u \sum_{j=m+1}^{n} (\lambda_j - \lambda^{\circ}_j)B_j x^{\circ} + Q_{\circ}N(\lambda)x^{1*}(\underline{\lambda}_m, \underline{\mu}_n, u)$$

$$-Q_{\circ} f((\underline{\lambda}_m, \underline{\mu}_n), ux + x(\underline{\lambda}^{\circ}_m, \underline{\mu}^{1*}_m, u)) = 0 \qquad (3.13)$$

Using the basis vectors $B_j x^{\circ}, i = 1, \ldots, m$ the bifurcation function

$F = (F,\dots,F_m) : \Re^m \times \Re^{n-m} \times \Re \rightarrow \Re^m$ is defined by

$$\sum_{j=1}^{m} F_j(\underline{\lambda}_m, \underline{\mu}_m, u)\ B_j x^\circ := -u \sum_{j=1}^{m} (\lambda_j - \lambda_j^\circ) B_j x^\circ - \sum_{j=1}^{m} G_j(\underline{\lambda}_m, \underline{\mu}_m, u).$$

where

$$\sum_{j=1}^{m} G_j(\underline{\lambda}_m, \underline{\mu}_m, u) B_j x^\circ := Q_\circ \{u \sum_{j=m+1}^{n} (\lambda_j^\circ - \lambda_j) B_j x^\circ - W(\lambda) x^{1*}(\underline{\lambda}_m, \underline{\mu}_m . u) + f((\underline{\lambda}_m, \underline{\mu}_m), u x^\circ + x^{1*}(\underline{\lambda}_m, \underline{\mu}_m, u)\}$$

thus

$$F(\underline{\lambda}_m, \underline{\mu}_m, u) = -u(\underline{\lambda}_m - \underline{\lambda}_m^\circ) - G(\underline{\lambda}_m, \underline{\mu}_m, u)$$

where

$$G = (G_1,\dots,G_m) : \Re^m * \Re^{n-m} * \Re \rightarrow \Re^m$$

satisfies

$$G(\underline{\lambda}_m, \underline{\mu}_m, 0) = 0,$$

so that

$$D^k_{\underline{\mu}_m} G(\underline{\lambda}_m, \underline{\mu}_m, 0) = 0, \quad 1 \le k \le r$$

and

$$D_u\ G(\underline{\lambda}_m^\circ, \underline{\mu}_m^\circ, 0) = 0.$$

It follows that

$$G(\underline{\lambda}_m, \underline{\mu}_m, u) = u \tilde{G}(\underline{\lambda}_m, \underline{\mu}_m, u)$$

where

$$\tilde{G} \in G^{r-1}(\Re^m * \Re^{n-m} * \Re, \Re^m).$$

and the bifurcation equations reduces to

$$H(\underline{\lambda}_m, \underline{\mu}_m, u) := -(\underline{\lambda}_m - \underline{\lambda}_m^\circ) - \tilde{G}(\underline{\lambda}_m, \underline{\mu}_m, 0) = 0$$

now,

$$\tilde{G}(\underline{\lambda}_m^\circ, \underline{\mu}_m^\circ, 0) = 0 \text{ and } D_{\underline{\lambda}_m} \tilde{G}(\underline{\lambda}_m^\circ, \underline{\mu}_m^\circ, 0) = 0$$

so that

$$H(\underline{\lambda}_m^\circ, \underline{\mu}_m^\circ, o) = 0 \text{ and } D_{\underline{\lambda}_m} H(\underline{\lambda}_m^\circ, \underline{\mu}_m^\circ, 0) = -Idm$$

where Idm denotes the identity mapping on $\Re^n$.

Therefore, by the implicit function theorem, there is a neighborhood VC $\Re^{n-m} \times \Re$ of $(\underline{\mu}_m^\circ, 0)$ and of and a unique function $\underline{\lambda}_m^* \in C^{r-1}(V, \Re^m)$ such that

$$H(\underline{\lambda}_m^*(\underline{\mu}_m, u)) = 0 \qquad \forall\ (\underline{\mu}_m, u) \in V$$

and

$$\underline{\lambda}_m^*(\underline{\mu}_m, u) = \underline{\mu}_m^\circ + 0\left(|u| + \left\| \underline{\mu}_m - \underline{\mu}_m^\circ \right\|\right)$$

The (3.5) has non-trivial solutions

$$(\underline{\lambda}_m^*(\underline{\mu}_m, u), \underline{\mu}_m), x^{1*}(\underline{\mu}_m, u) \in R^n \times X : (\underline{\mu}_m, u) \in V$$

where

$$\underline{\lambda}_m^*(\underline{\mu}_m, u) = \underline{\lambda}_m^\circ + 0\left(|u| + \left\| \underline{\mu}_m - \underline{\mu}_m^\circ \right\|\right)$$

and

$$x^{1*}(\underline{\mu}_m, u) = ux^\circ + 0\left(|u|^2 + |u| \left\| \underline{\mu}_m - \underline{\mu}_m^\circ \right\|\right)$$

as $\left\| \underline{\mu}_m - \underline{\mu}_m^\circ \right\|, |u| \to 0.$

We also have the following result on linear perturbation of the system (1.1) and the proof is again as Theorem 3.2.

THEOREM 3.3

Suppose H_i, $i = 1, \dots, k$ are Helpert spaces $A_i(\underline{\varepsilon}), B_{ij}(\underline{\varepsilon}) : H_i \to H_i$ are bounded self-adjoint linear operators continues in $\underline{\varepsilon}$(or C^r resp analytic), $\underline{\varepsilon} \in C^l$, for $\underline{\varepsilon}$ in an open neighborhood of $0 \in C^l$. If $\underline{\lambda} = \underline{\lambda}^\circ \in \Re^n$ is such that each of the proplems in system (3.3) has a solution $x_i^\circ \neq 0$ and the conditions of LEMMA(3.1) are satisfied, then there is a constant $\delta > 0$ such that for $\underline{\varepsilon} \in C^1$, $\|\underline{\varepsilon}\| < \delta$, $\|\underline{\mu}_m - \underline{\mu}^\circ\| < \delta$ there is a bifurcation i.e solutions $((\underline{\lambda}^*_m(\varepsilon, \underline{\mu}_m), \underline{\mu}_m), x^*(\underline{\varepsilon}, \underline{\mu}_m))$ of the system

$$(A_i(\varepsilon) - \sum_{j=1}^{n} \lambda_j A_{ij}(\varepsilon))x_i = 0 \qquad i = 1, \ldots, m$$

with $(\lambda^*_m(0, \underline{\mu}^\circ), \underline{\mu}^\circ_m) = (\underline{\lambda}^\circ_m, \underline{\mu}^\circ_m) = \underline{\lambda}^\circ, x^*(0, \underline{\mu}^\circ_m) = x^\circ$

The functions $\underline{\lambda}^*_m(\underline{\varepsilon}, \underline{\mu}_m), x^*(\underline{\varepsilon}, \underline{\mu}_m)$ are continuus in $\underline{\varepsilon}$(or C^r resp analytic).

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تعميم القيم الذاتيه البسيطه والتشعب لمساله قيم ذاتيه مرتبطه متعدده البارامترات

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الملخص العربى

فى هذا البحث تم دراسه مجموعة من المعادلات الغير خطية متعدده البارمترات والمرتبطة مع بعضها بهذه البارمترات والتى تحتوى على مؤثرات متوافقة ذاتيا (فى حاله اعتمادها او عدم اعتمادها تحليلا على بارامتر اضافى) حيث عدد المعادلات اقل من عدد البارمترات وبفرض ان بعد نواه الموثر الخطى هو الوحده ثم اثبات انه لهذه المجموعه الغير خطيه حل متشعب من الحل الصفرى عند القيمه الذاتيه البسيطه للمؤثر الخطى.