Mansoura University
Faculty of Engineering
Math. & Eng. Physics. Dept.

Functional Analysis

1st year, Postgraduate studies.
Final Exam, Sep. 2013
Time Allowed: 3 hour

Question (1): Sate whether the following statements are true or false

a) Every countable set is finite.

b) A convergent sequence has a unique limit.

c) The differential operator is a bounded operator.

d) Every subspace is a convex set and vice versa.

e) In a finite dimensional normed space, every linear operator is bounded.

f) All normed spaces are inner product spaces.

g) A contraction mapping is a continuous mapping.

Question (2): Define the following terms

a) The supremum of a bounded above set M.

b) Isometric mapping.

c) Open set.

d) Accumulation point.

e) Strict convexity.

f) Dual space.

g) Equivalent norm.

h) Total set.

i)Best approximation.

Question (3):

a) Is the subset of all x, $x = (a_1, a_2, a_3)$ with positive a_1, a_2 and negative a_3 constitute a subspace of R^3? .

b) If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$, converges.

Question (4):

a) If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

b) Show that every finite dimensional subspace Y of a normed space X is complete.

c) If the composite of two linear operator exists, show it is linear.

Question (5):

a) What are the main advantages of orthonormal sequences over arbitrary linearly independent sequences?

b) Let X be an inner product space. Prove that for all $x, y, and\ z \in X$

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2.$$

c) For the operator $L=a_0(x)\dfrac{d^2}{dx^2}+a_1(x)\dfrac{d}{dx}+a_2(x)$, find L^* where $Bu=\begin{bmatrix}0\\0\end{bmatrix}$ and $B^*u=\begin{bmatrix}0\\0\end{bmatrix}$.

Question (6):

a) Consider Fredholm integral equation $x(t)-\mu\int_a^b k(t,\tau)x(\tau)d\tau=v(t)$. Deduce the condition for this equation to have a fixed point according to Banach fixed point theorem.

b) Prove that in a normed space $(X,\|\ \|)$ the set M of best approximations to a given point x out of a subspace Y of X is convex.