Sci. J. Fac. Sci. Minufia Univ., 2007, Vol. XXI, 9-44

ON STABILITY OF THE METHOD OF LINES

AMR A. SHARAF AND H.O. BAKODAH*
Dept. computer Science , faculty of science , King Abdul Aziz University ,Jeddah
amrsh @ aucegypt.edu

* Dept. of Mathematics, Faculty of Girls , Jeddah , K.S.A.
hobak @ gawab.com

ABSTRACT

In this paper, The stability analysis for the method of lines (MOL) will be established analytically and computationally for both initial and boundary value problem in ordinary differential systems.

1- INTRODUCTION

The stability analysis constitute the essential study of the numerical solution of partial differential equations in general this is because such study provides the means by which the step size and the numerical integration scheme for the given differential equation could be selected so as to secure manageable numerical solution .

Regarding the method of lines for parabolic, hyperbolic ,and elliptic partial differential equation (in two variables),they can be classified according to the nature of the resulting system in connection with the direction of daiscretization as shown in the following table in which some examples will be illustrated :

Table (1)
Nature of the system in connection with the direction of discretization

EXAMPLES FOR PARTIAL DIFFERENTIAL EQUATIONS IN TWO VARIABLE	DISCRETIZATION DIRECTION	NATURE OF THE RESULTING SYSTEM
Parabolic	X - direction	Initial value type in ODE.
Parabolic	T - direction	Boundary value type in ODE.
Hyperbolic	X – direction	Initial value type in ODE.
Hyperbolic	T – direction	Boundary value type in ODE.
Elliptic	Y (or X)- direction	Boundary value type in ODE.

The stability analysis of the method of lines for discretization that produce initial value type in ordinary differential equations is well studied by various authors and will be the subject of section 2 and section 3of the present paper .

In view of the too complicated behavior of the boundary value problems than that of the initial value one, it is not surprising no serious attempts except that of Jones et al. [12] have been made to analyze the stability of the method of lines for discretization that produce boundary valued problems in ordinary differential equations consequently a study must now be developed to fill the gap left for the stability analysis of the method of lines for:

(1) The elliptic partial differential equations.

(2) The parabolic and hyperbolic partial differential equations under discretization that produce boundary value system .

In section **4** a technique for the stability analysis to elliptic differential equations when treated by the method of lines will be established.

Consequently, a general and flexible computational algorithm may now be devised once for all types of partial differential equations. Such computational algorithm will be considered in section **5**.

2- STABILITY FOR PARABOLIC EQUATIONS

In order to analyze the stability of parabolic equations we consider the heat equations as a typical example

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial x^2} \qquad (2.1)$$

Without loss of generality, discussion is limited to equation (2.1) in region $-1 \le x \le 1$.

Boundary conditions are:

$$\begin{aligned} &\bar{u}(-1,t) = a \ ,t > 0 \\ &\bar{u}(1,t) = b \quad ,t > 0 \\ &\bar{u}(x,0) = 0 \qquad , \quad -1 < x < 1 \end{aligned} \qquad (2.2)$$

It is convenient to restate the problem as follows:

Let $\bar{u} = u + \left[\frac{(b-a)}{2}\right] x + \frac{a+b}{2}$; then (2.1) becomes:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (2.3)$$

and (2.2) become :

$$\begin{aligned} &u(-1,t) = 0 \ ,t > 0 \\ &u(\ 1,t) \ = 0 \ ,t > 0 \\ &u(x,0) = \frac{-(b-a)\,x}{2} - \frac{(a+b)}{2} \quad , \quad -1 < x < 1 \end{aligned} \qquad (2.4)$$

By discretization (2.3) with respect to x , the following system of ordinary differential equations is obtained :

$$\frac{du_i}{dt} = \sum a_{ij} \; u_j \; . \qquad (2.5)$$

Where $\sum a_{ij} \; u_j$ is a finite difference approximation to $\frac{\partial^2 u}{\partial x^2}$ at the discrete point x_j .

Specifically , if the finite difference approximation to $\frac{\partial^2 u}{\partial x^2}$ involves "*2K+1*" points centered about u_i , and the closed interval [-1,1] is divided into $2N$ equal intervals , this equation (2.5) becomes :

$$\frac{du_j}{dt} = \sum_{i=j-k}^{j+k} a_{ij} u_i \qquad , \; j = -(N-1), -(N-2), \ldots, -1, 0, 1, \ldots, (N-1) \qquad (2.6)$$

a total of *2N-1* equations in *2N-1* dependent variables . If the central difference approximation is to be used throughout, assumptions must be made about the values of *k-l* points outside each end of the interval . In dealing with 5 point central difference approximations to the second partial derivatives, Fisher [5] proposed that , at the endpoints, where u_N ,, $\frac{du_N}{dt}$, $\frac{d^2 u_N}{dt^2}$, all equal zero because of (2.4), the 3 point central
difference approximation also can be assumed valid . This assumption leads to the requirement that $u_{N+1} = -u_{N-1}$ and $u_{-(N+1)} = -u_{-(N-1)}$. All equally spaced central difference approximations to the second derivative are symmetric in the values of the coefficients of u_i about the central point . Therefore, if Fisher's proposal is extended to require that , staring with the 3 point formula , all odd point central difference formulas for the second partial derivative, up to and including the odd point formula being used , agree in being equal to zero at the endpoints, then

$$u_{N+k} = -u_{N-k} \; , u_{-(N+k)} = -u_{-(N-k)} \; , \; k \geq 1$$

This is a specification of the required values of outside the endpoints. This specification does not violate common sense , and is also in accord with the Fourier series solution of (2.3) subject to (2.4) where such antisymmetry occurs about the endpoints of the interval . It should be noted also that where is no reason to use an even number of points for approximating the second derivative because the next lower odd point central difference formula always gives an approximation of the same order.

Equations (2.6) for a 3 point central difference approximation are , in matrix form

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 0 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & 1 \\ 0 & . & . & . & \cdots & 1 & -2 \end{bmatrix}\begin{bmatrix} u_{N-1} \\ \\ . \\ . \\ . \\ \\ u_{-(N-1)} \end{bmatrix} = \begin{bmatrix} \frac{du_{N-1}}{dt} \\ . \\ . \\ . \\ \frac{du_{-(N-1)}}{dt} \end{bmatrix} \tag{2.7}$$

or $\frac{1}{h^2}A_3 u = \frac{d}{dt}u$ where $h = \frac{1}{N}$.

The solution is $u = \sum_{k=1}^{2N-1} C_k\, E_k \exp(\frac{\lambda_k t}{h^2})$, where the λ_k are the eigenvalues of A_3, E_k are eigenvectors of A_3 , and the C_k are the Fourier coefficients of (2.5) .

The eigenvalues of A_3 are given by

$$\lambda_k = -2 + 2\cos\frac{k\pi}{2N} \quad , \; k = 1,2,\ldots,2N-1$$

Normalized eigenvectors are found, by direct calculation, to be:

$$\sqrt{\tfrac{1}{n}}\begin{bmatrix} \sin\frac{k\pi}{2N} \\ \sin\frac{2k\pi}{2N} \\ \sin\frac{3k\pi}{2N} \\ . \\ . \\ . \\ \sin\frac{(2N-1)\,k\pi}{2N} \end{bmatrix} \quad , \; k = 1,2,\ldots,2N-1 \tag{2.8}$$

These eigenvectors have a trigonometric character . These are appropriate functions for fitting the boundary conditions (2.4) . They are, in fact eigenfunctions of (2.3) at the discrete points x_i .

Equations (2.6) for a 5 point central difference approximation are , in matrix form,

$$\frac{1}{12h^2}\begin{bmatrix} -29 & 16 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 16 & -30 & 16 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 16 & -30 & 16 & -1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & & . & . & . \\ . & . & . & . & \dots & & . & . & . \\ . & . & . & . & \dots & & . & . & . \\ 0 & . & . & . & \dots & & -1 & 16 & -29 \end{bmatrix}\begin{bmatrix} u_{N-1} \\ \\ . \\ . \\ . \\ \\ u_{-(N-1)} \end{bmatrix} = \begin{bmatrix} \frac{du_{N-1}}{dt} \\ \\ . \\ . \\ . \\ \\ \frac{du_{-(N-1)}}{dt} \end{bmatrix}$$

or

$$\frac{1}{12h^2}\mathrm{A}_5\, u = \frac{d}{dt}u \qquad (2.9)$$

In general , for an " n " point finite difference approximation

$$\frac{2}{(n-1)!h^2}\begin{bmatrix} \textit{Cofficients of the} \\ \textit{Lagrangian Formulas} \\ \textit{for numerical} \\ \textit{Differentiation} \end{bmatrix}\begin{bmatrix} u_{N-1} \\ \\ . \\ . \\ . \\ \\ u_{-(N-1)} \end{bmatrix} = \begin{bmatrix} \frac{du_{N-1}}{dt} \\ \\ . \\ . \\ . \\ \\ \frac{du_{-(N-1)}}{dt} \end{bmatrix}$$

Or

$$\frac{2}{(n-1)!\,h^2}\mathrm{A}_n\, u = \frac{d}{dt}u \qquad (2.10)$$

Now it can be shown , by direct calculation (for n = 5,7,9,11) , that the recursion formula :

$$\mathrm{A}_n = (-1)^{\frac{(n+1)}{2}}\left[\left(\tfrac{n-3}{2}\right)!\right]^2 \mathrm{A}_3^{\frac{(n-1)}{2}} + (n-1)\,(n-2)\,\mathrm{A}_{n-2} \qquad (2.11)$$

holds . The significance of (2.11) is that each of the matrices, A_n, is a polynomial in A_3 , and , therefore , commutes with A_3,which implies that A_n has the same eigenvectors as A_3 . Now the eigenvalues of A_3 are real and negative . Formula (2.11) can also be used to compute the eigenvalues of A_n, which are always real and negative,and which approach the eigenvalues of (2.3) with increasing N , as shown by Fisher[5] .

Therefore , there is stability and also convergence to the true solution with increasing N .Used of these eigenvectors to approximate the function $u(x,0)$ of (2.4) is equivalent to fitting $u(x,0)$ at discrete points by trigonometric interpolation. If $u(x,0)$ has a conve- rgent Fourier series expansion , the discrete approximations will converge to $u(x,0)$ as N increases . Any stable , convergent numerical algorithm applied to solving resulting system of ordinary differential equations will then also produce a stable , convergent numerical solution of the associated partial differential equation .

Whether or not the use of the higher order difference approximations will improve convergence depends on the importance of the higher eigenvalues . As the number of points "n" used to approximate is increased (keeping N constant), the eigenvalues $\frac{2\lambda k}{[(n-1)!\,h^2]}$ of the approximating system of ordinary differential equations. (2.6) more and more closely approach $\frac{-k^2\pi^2}{4}$, the eigenvalues appearing in the solution of (2.3) by Fourier series . Thus, as N is increased , the solution of system (2.6) approaches a truncation of the Fourier series solution of (2.3) to $2N-1$ terms . If the first three or four terms of the series adequately determine the solution , then use of higher order differences will not be necessary .

It is noted above that the use of a central difference approximation of order greater than 3 .requires explicit specification of dependent variable values outside the interval of interest . Explicit consideration at either end of the x interval. For example, with a 5 point difference approximation, instead of the central difference formulation

$$\frac{\partial^2 y_{n-1}}{\partial x^2} \cong \frac{1}{12h^2}(-y_{n-3}+16y_{n-2}-30y_{n-1}+16y_n-y_{n+1})$$

where y_{n+1} must be specified in terms of the other y's , we can use (see Kopal [14])

$$\frac{\partial^2 y_{n-1}}{\partial x^2} \cong \frac{1}{12h^2}(-y_{n-4}+4y_{n-3}+6y_{n-2}-20y_{\omega-1}+11y_{\omega})$$

which does not involve points outside of $-1 \le x \le 1$ .Numerical experimentation[7,10] shows this approach to be significantly more than use of the central difference formul- ation when eigenvalues are not important .The eigenvalues, for a 5 point noncentral difference approximation to $\frac{\partial^2 u}{\partial x^2}$ are real and negative .This assures stability .

Theorem : the eigenvalues of the non-central difference matrix B , given by

$$B = \begin{bmatrix} -20 & 6 & 4 & -1 & 0 & 0 & \cdots & 0 \\ 16 & -30 & 16 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 16 & -30 & 16 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\cdot \\ \cdot & \cdot & \cdot & 0 & -1 & 16 & -30 & 16 \\ \cdot & \cdot & \cdot & 0 & -1 & 4 & 6 & -20 \end{bmatrix} \qquad (2.12)$$

are all real , negative , and lie between the eigenvalues of the Fisher matrix F , where

$$F = \begin{bmatrix} -29 & 16 & -1 & 0 & \cdots & 0 \\ \multicolumn{6}{l}{same\ as\ 2nd\ to\ (n-1)st\ rows\ \ of\ \ B} \\ 0 & \cdots & & -1 & 16 & -29 \end{bmatrix} \qquad (2.13)$$

Proof:

We use the equations (2.12) ana (2.13) , we obtain

$$F+\begin{bmatrix} 9 & -10 & 5 & -1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ 0 & \cdots & & 0 & -1 & 5 & -10 & 9 \end{bmatrix} = B \qquad (2.14)$$

If n is the dimension of F ,then E ,the normalized eigenvector matrix of F ,is given by

$$E=\sqrt{2/(n+1)}\begin{bmatrix} \sin\frac{\pi}{n+1} & \sin\frac{2\pi}{n+1} & \cdots & \sin\frac{\pi}{n+1} \\ \sin\frac{2\pi}{n+1} & \sin\frac{4\pi}{n+1} & \cdots & \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \\ \sin\frac{\pi}{n+1} & -\sin\frac{2\pi}{n+1} & \cdots & (-1)^n\sin\frac{\pi}{n+1} \end{bmatrix} \qquad (2.15)$$

By using the notation $\alpha_i = (\frac{4}{n+1})\sin\frac{i\pi}{n+1}$ and

$$\beta_j = \left(9\sin\frac{j\pi}{n+1} - 10\sin\frac{2j\pi}{n+1} + 5\sin\frac{3j\pi}{n+1} - \sin\frac{4j\pi}{n+1}\right)$$

the matrix $E^{-1}BE$ can be written as

$$E^{-1}BE = \begin{bmatrix} \alpha_1\beta_1+\lambda_1 & 0 & \alpha_1\beta_3 & \cdots \\ 0 & \alpha_2\beta_2+\lambda_2 & 0 & \cdots \\ \alpha_3\beta_1 & 0 & \alpha_3\beta_3+\lambda_3 & \\ \cdots & & & \\ \cdots & & & \\ \cdots & & & \end{bmatrix}$$

The eigenvalues of $E^{-1}BE$ (same as B) are given by the roots of a determinantal equation . By row and column interchanges and some simple algebraic manipulations , the determinantal equation can be expressed in polynomial form as

$$\left[\prod_{i\ odd}\left(\frac{\lambda_i - x}{\alpha_i}\right) + \sum_{j\ odd}\beta_j \prod_{\substack{i\ odd\\ i\neq j}}\left(\frac{\lambda_i - x}{\alpha_i}\right)\right]\cdot\left[\prod_{i\ even}\left(\frac{\lambda_i - x}{\alpha_i}\right) + \sum_{j\ even}\beta_j \prod_{\substack{i\ odd\\ i\neq j}}\left(\frac{\lambda_i - x}{\alpha_i}\right)\right] = 0 \quad (2.17)$$

If $B_k = 0$, then $x = \lambda_k$ is a root , and we factor $(\lambda_k - x)$ out of (2.17) . Then by dividing each of the above by its first term , we obtain (let $m_i = \alpha_i \beta_i$)

$$\left[1 + \sum_{i\ odd}\frac{m_i}{\lambda_i - x}\right]\left[1 + \sum_{i\ even}\frac{m_i}{\lambda_i - x}\right] = 0 \qquad (2.18)$$

where $\lambda_1 > \lambda_2 > \ \dots\ > \lambda_n$ because $\lambda_i = -28 + 32\cos\frac{i\pi}{n+1} - 4\cos^2\frac{i\pi}{n+1}$.

Let $f_1(x) = 1 + \sum_{i\ odd}\frac{m_i}{\lambda_i - x}$ and $f_2(x) = 1 + \sum_{i\ even}\frac{m_i}{\lambda_i - x}$, now if $m_i\ m_{i+2} > 0$, it follows consideration of the singularities of (2.18) that there is a root of (2.18) (and of (2.17))between λ_i and λ_{i+2} . Now , by using the appropriate trigonometric identities (let $z_i = \cos\frac{i\pi}{n+1}$) :

$$m_i = \frac{16}{n+1}(1 - z_i^2)\ (1 - z_i)^2(1 - 2z_i) \qquad (2.19)$$

Because m_i is positive for $-1 < z_i < \frac{1}{2}$, m_i is positive for $\frac{n+1}{3} < i \leq n$. If $n+1$ is divisible by 3 , $m_{\frac{n+1}{3}} = 0$.

Because the λ_i are strictly decreasing with i , we have only to consider (2.18) with one change of sign between adjacent m's in the first term, and at most one change of sign between adjacent m's in the second term . We know at once , then , that there are at least $n-4$ negative real roots to (2.17),each lying between two eigenvalues of F .

The eigenvalues of F are given by

$$\lambda_i = -28 + 32\cos\frac{i\pi}{n+1} - 4\cos^2\frac{i\pi}{n+1} \qquad (2.20)$$

For $i=\dfrac{n+1}{3}$, $\lambda_i=-13$, we show that

$$f_1(-13)=1-\sum_{i\ odd}\left|\frac{m_i}{\lambda_i+13}\right|>0 \tag{2.21a}$$

and

$$f_2(-13)=1-\sum_{i\ even}\left|\frac{m_i}{\lambda_i+13}\right|>0 \tag{2.21b}$$

and thus prove the theorem .

We show below that , in fact

$$-\sum_{i=1}^{n}\left|\frac{m_i}{\lambda_i+13}\right|>0 \tag{2.22}$$

This is sufficient condition for $f_1(13)>0$ and $f_2(13)>0$.

$$\left|\frac{m_i}{\lambda_i+13}\right|=\left|\frac{\frac{16}{n+1}(1-z_i^2)\,(1-z_i)^2(1-2z_i)}{-15+32z_i-4z_i^2}\right|$$

$$=\frac{16}{n+1}\frac{(1+z_i)\,(1-z_i)^3}{(15-2z_i)}$$

or

$$\left|\frac{m_i}{\lambda_i+13}\right|\le\frac{16}{n+1}\frac{(1+\cos\frac{i\pi}{n+1})(1-\cos\frac{i\pi}{n+1})^3}{(15-2\cos\frac{i\pi}{n+1})}$$

so if

$$\left|\frac{m_i}{\lambda_i+13}\right|\le\frac{16}{n+1}\frac{(1+\cos\frac{i\pi}{n+1})(1-\cos\frac{i\pi}{n+1})}{13}$$

so

$$\frac{16}{n+1}\sum_{i=1}^{n}\frac{(1-2\cos\frac{i\pi}{n+1}+2\cos^3\frac{i\pi}{n+1}-\cos^4\frac{i\pi}{n+1})}{13}\le 1$$

the theorem is proved.

From the following relations , see [11]

$$\sum_{i=1}^{n} \cos\frac{i\pi}{n+1} = 0 = \sum_{i=1}^{n} \cos^3\frac{i\pi}{n+1} \qquad \text{by symmetry}$$

$$\sum_{i=1}^{n} \cos^4\frac{i\pi}{n+1} = \frac{1}{8}(3n-5) \qquad \text{by symmetry}$$

therefore

$$\frac{16}{n+1}\sum_{i=1}^{n} \frac{(1-2\cos\frac{i\pi}{n+1}+2\cos^3\frac{i\pi}{n+1}-\cos^4\frac{i\pi}{n+1})}{13} = \frac{16}{n+1}\,\frac{8n-3n+5}{8\times 13} = \frac{10}{13} < 1$$

Q.E.D.

An alternative formulation of an $O(h^4)$ method, one [19] uses the Numerov finite difference formula, this is based on replacing the right hand side of (2.3) by

$$\frac{1}{12}\left(\frac{\partial^2 u}{\partial x^2}(x_i - h, t) + 10\frac{\partial^2 u}{\partial x^2}(x_i, t) + \frac{\partial^2 u}{\partial x^2}(x_i + h, t)\right)$$

which leads to the $O(h^4)$ system (for "3" point scheme),

$$(u_{i+1} - 2u_i + u_{i-1})/h^2 = \frac{1}{12}\left(\frac{du_{i+1}}{dt} + 10\frac{du_i}{dt} + \frac{du_{i-1}}{dt}\right) \tag{2.23}$$

which can be shown to posses real and negative eigenvalues, then the system is stable.

3- STABILITY FOR HYPERBOLIC EQUATIONS

In order to study the stability of hyperbolic equations, first we consider the first order equations ,and ,then, we consider the second order equations, taking the wave equation as atypical example.

3.1 First order equations:

The method presented in this subsection uses a three point difference scheme which is a biased average of forward and backward differences. The direction and amount of the bias are adjusted to give stable difference schemes with as accuracy between the usual first and second order schemes.

We first study the accuracy of the following difference scheme [6]

$$\frac{df_i}{dx} = \frac{(A f_{i+1} + C f_i - B f_{i-1})}{(A+B)\,\Delta x} + E_T \tag{3.1}$$

and

$$C = B - A = \pm 1 \qquad (3.2)$$

where A and B are integers greater than or equal to zero , $A+B$ is an odd integer , and E_T is the truncation error . This equation may be rearranged to give

$$\frac{df_i}{dx} = \frac{1}{A+B}\left[A\left(\frac{f_{i+1}-f_i}{\Delta x}\right)+B\left(\frac{f_i - f_{i-1}}{\Delta x}\right)\right]+E_T \qquad (3.3)$$

This arrangement shows that the proposed scheme is a weighted average of A forward differences and B backward differences . By replacing the two differences with their Taylor series expansions , one obtains

$$\frac{df_i}{dx} = \frac{A}{A+B}\left[\frac{df_i}{dx}+\frac{\Delta x}{2}\frac{d^2 f_i}{dx^2}+\frac{(\Delta x)^2}{6}\frac{d^3 f_i}{dx^3}+ \;\dots\right]$$

$$+\frac{B}{A+B}\left[\frac{df_i}{dx}+\frac{\Delta x}{2}\frac{d^2 f_i}{dx^2}+\frac{(\Delta x)^2}{6}\frac{d^3 f_i}{dx^3}+ \;\dots\right]+E_T \qquad (3.4)$$

Since $B-A=\pm 1$, this equation reduces to

$$E_T = \pm\frac{1}{A+B}\frac{\Delta x}{2}\frac{d^2 f_i}{dx^2}+\frac{(\Delta x)^2}{6}\frac{d^3 f_i}{dx^3} \pm \;\dots \qquad (3.5)$$

For $A+B$ equal to 1 , the difference scheme reduces to either the usual forward or backward differences with its associated first order truncation error . When $A+B$ is greater than 1, the first –order term of the truncation error is reduced by a factor of $A+B$. In the limit as $A+B$ approaches infinity ,this differences scheme approaches the second order accuracy of a central difference .

Now the stability to be shown to the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial u}{\partial x} \qquad (3.6)$$

We use the scheme (3.1) one can obtain :

$$\frac{du_i(t)}{dt} = D\frac{(A\,u_{i+1}+C\,u_i - B\,u_{i-1})}{(A+B)\,\Delta x} \qquad (3.7)$$

where $u(t)$ is the approximate value of u at x_i and t . We use implicit Euler scheme for solving (3.7) , then the resulting system is

$$\frac{(u_i^{n+1}-u_i^n)}{\Delta t}=D\,\frac{(A\,u^{n+1}{}_{i+1}+C\,u_i^{n+1}-B\,u^{n+1}{}_{i-1})}{(A+B)\,\Delta x}+ \tag{3.8}$$

where $u^n{}_i(t)$ the solution to the difference equation at x_i and t_n. To show unconditional stability with Fourier method of analysis [18], one assumes that the solution of the difference equation is of the form

$$U_i^n=\gamma^n\; e^{j\,p_i\,\Delta x} \tag{3.9}$$

where $j=(-1)^{\frac{1}{2}}$ and then shows that the magnitude of the complex constant γ is less than 1 (the von Neumann condition). Substituting this expression for U_i^n into equation (3.8) and simplifying lead to

$$\gamma=\left(1-rC-rA\,e^{jp\,\Delta x}+rB\,e^{-jp\,\Delta x}\right)^{-1} \tag{3.10}$$

where

$$r=\frac{D\,\Delta t}{(A+B)\,\Delta x} \tag{3.11}$$

The equation (3.10) may be rewritten as

$$\gamma=\left(1-rC(1-\cos p\,\Delta x)-r\,j\,(A+B)\sin p\,\Delta x\,\right)^{-1} \tag{3.12}$$

For this procedure(equation(3.8))to be unconditionally stable, the magnitude of γ must be lees than 1.

The requirement is satisfied if

$$1-rC(1-\cos p\,\Delta x\,)>1 \tag{3.13}$$

(using the fact that $z=\frac{1}{a-ib}$ i.e. $z=\frac{1}{a}\left\{\frac{1+i\frac{b}{a}}{1+\frac{b^2}{a^2}}\right\}$ is less than 1 if $a>1$). This condition may also be written as

$$C\left(\frac{D\,\Delta t}{(A+B)\,\Delta x}\right)(1-\cos p\,\Delta x<0 \tag{3.14}$$

Since $\cos p\,\Delta x\le 1$, C must be positive if D is negative and C must be negative if D is positive. This implies that $A<B$ for $-D<0$ and $A>B$ for $D<0$. To maintain stability , the averaging of the forward and backward differences based upstream relative to the motion of the wave . As *(A+B)* increases in value the magnitude of γ approaches 1, which is the condition of marginal stability .

For an explicit time procedure (Euler scheme), the finite difference scheme for equation (3.7) is given by

$$\frac{(u_i^{n+1} - u_i^n)}{\Delta t} = D \, \frac{(A \, u^n{}_{i+1} + C \, u_i{}^n - B \, u^n{}_{i-1})}{(A + B) \, \Delta x} + \qquad (3.15)$$

Using the same steps as in implicit scheme , we find the condition $|r| < 1$ leads to the restriction

$$\Delta t \leq \frac{-C \, \Delta x}{D \, (A + B)} \qquad (3.16)$$

on the size of the time step . Since Δx and Δt are both positive , this equation gives the relationship between the signs of C and D as that found for the implicit case .

For $(A+B)$ equal to 1 , equation (3.16) gives the time step restriction for an explicit procedure based on forward $D > 0$ or backward $D < 0$ differencing . As $A+B$ increases , the time step restriction appears to become more severe. However, since the spatial truncation error is reduced by a factor of $A+B$, a value of Δx that is $A+B$ time as large can be used with biased difference scheme for the same spatial truncation error . Therefore a given error, the maximum time step allowed for stability of the biased difference scheme is independent of the value of $A+B$ as long as the first order term dominates the truncation error . since a larger value of Δx is used with the biased difference scheme ,fewer difference equations must be solved per time step . In reference[8], the authors employ the above technique using the package for the solution of ordinary differential equations by Hindmarsh [10] .

We are concerned , now , with the application of a few of the standard methods for ordinary differential equations to equation (3.6) , with $D = -1$. First , we describe the scheme for ordinary differential equations . we write the equation in the form

$$\dot{x} = f(x,t) \qquad (3.17)$$

and denote the approximation to $y(t_n) = y(n \, \Delta t)$ by y_n . Then the leapfrog scheme is

$$y_{n+1} = y_{n-1} + 2 \, \Delta t \, f(y_n, t_n) \qquad (3.18)$$

The Runge-Kutta version

$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \qquad (3.19)$$

where

$$k_1 = \Delta t\ f(y_n, t_n)$$
$$k_2 = \Delta t\ f(y_n + \tfrac{k_1}{2}, t_n + \tfrac{\Delta t}{2})$$
$$k_3 = \Delta t\ f(y_n + \tfrac{k_2}{2}, t_n + \tfrac{\Delta t}{2})$$
$$k_4 = \Delta t\ f(y_n + k_3, t_{n-1})$$

The Milne corrector is based on

$$y_{n+1} = y_{n-1} + \Delta t\ (f(y_{n-1}, t_{n-1}) + 4f(y_n, t_n) + f(y_{n+1}, t_{n+1})\ )\ /3 \qquad (3.20)$$

The other corrector is based on the following implicit Runge-Kutta scheme

$$y_{n+\frac{1}{2}} = \frac{(y_n + y_{n+1})}{2} + \Delta t\ \frac{(f(y_n, t_n) - f(y_{n+1}, t_{n+1}))}{8}$$

$$y_{n+1} = y_n + \Delta t\ \frac{\left(f(y_n, t_n) + 4f(y_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + f(y_{n+1}, t_{n+1})\right)}{6} \qquad (3.21)$$

This scheme , which is implicit in $y_{n+\frac{1}{2}}$ and y_{n+1} has been applied to diffusion problems by Watanabe and Flood [2ٍ٤].

Adames predictor which has third order accuracy is

$$y_{n+1} = y_n + \Delta t\ \frac{(23f(y_n, t_n) - 16f(y_{n-1}, t_{n-1}) + 5f(y_{n+2}, t_{n+2})\ )}{12} \qquad (3.22)$$

These schemes are applied to equation (3.6) with $D = -1$. With fourth order difference approximation to $\frac{\partial u}{\partial x}$ namely ,

$$\delta_x(u) = \frac{(u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2})}{(12\ \Delta x)} \qquad (3.23)$$

This defines a system of ordinary differential equations which approximate (3.6)

$$\frac{du_i}{dt} = f_i(\underline{u}, t)\ \ ,\ i = 1(1)N \qquad (3.24)$$

where for equation (3.6) with $D = -1$

$$f_i = -\delta_x(\underline{u})_i$$

For a fixed ratio $\lambda = \frac{\Delta t}{\Delta x}$, the scheme defined above are subject to a stability restriction , By making the substitution .
$u_i^n = \gamma^n \exp(j\ p_n\ \Delta x)$

and imposing the restriction that γ^n should be bounded , we obtain the stability restriction on the ratio λ . These are
1-Leap frog : $\lambda \le 0.73$.
2-Runge Kutta : $\lambda \le 2.0$.
3- Milne implicit : $\lambda \le 1.26$.
4-Implicit Runge kutta : Unconditional stability , $\lambda < \infty$.
5-Adams predicator : $\lambda \le 0.5$.

The condition for the Adams predicator was determined numerically using a root finder .The same result is obtained from the stability regions given by Shampine and Gordan [20] .

Such schemes had been used by Miller [16] for many problems with graphical determination of the optimal mesh ratio λ and a comparison of the schemes .

Conditions that Runge-Kutta methods are locally stable when applied to numerical solution of hyperbolic partial differential equations are derived by Kreiss and scherer in [15], other result in [3,4,17].

For nonlinear problems, Strikwerda [٢٢] using the concept of well posed problem , to establish the stability, with the aid of the results in [٧] for finite difference method .

3.2 Second order problems:

In order analyze the stability , we consider the wave equation as a test problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le 1 \quad , \quad t \ge 0 \qquad (3.25a)$$

with the boundary conditions .

$$u(0,t) = u(1,t) = 0 \qquad (3.25b)$$

and the initial conditions

$$u(x,0) = f(x) \ , \ \frac{d}{dt}u(x,0) = g(x) \qquad (3.25c)$$

Using the standard "3" point central difference scheme to approximate the second derivative with respect to x variable, we obtain:

$$\frac{d^2u_i}{dx^2} = \frac{(u_{i+1} - 2u_i + u_{i-1})}{h^2} \quad , i = 1(1)N \qquad (3.26a)$$

where $h = \dfrac{1}{N+1}$, with

$$u_i(0) = f(x_i) = f_i \quad , i = 1(1)N \qquad (3.26b)$$

$$u_i'(0) = g(x_i) = g_i \quad , i = 1(1)N \qquad (3.26c)$$

$$u_0(t) = u_{N+1}(t) = 0 \qquad (3.26d)$$

let $v_i = \dfrac{du_i}{dt}$, equation (3.26a) can be written in the matrix vector form

$$\underline{\dot{u}}(t) = \underline{v}(t) \quad , \quad \underline{u}(0) = \underline{f} \qquad (3.27a)$$

and

$$\underline{\dot{v}}(t) = A_1\, \underline{u}(t) \quad , \quad \underline{v}(0) = \underline{g} \qquad (3.27b)$$

where A_1 is a matrix of $N \times N$ with

$$a_{ij} = \begin{cases} 2 & , i = 1 \, , j = 2 \\ -2 & , i = j \\ 1 & , |i - j| = 1 \ , i \neq j \\ 0 & , \ othewise \end{cases}$$

$\underline{f}$ is a vector of i^{th} component is f_i , and $\underline{g}$ is a vector of i^{th} component is g_i.

One can define the vector W with 2 N component, as follows

$$W(t) = \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix}$$

then we can rewrite equation (3.27) as follows

$$\dot{W}(t) = A\ W(t) \quad , \quad W(0) = \begin{bmatrix} \underline{f} \\ \underline{g} \end{bmatrix} \qquad (3.28)$$

where A is an $2N \times 2N$ matrix and is defined as follows

$$A = \begin{bmatrix} \mathrm{O} & \mathrm{I} \\ A_1 & \mathrm{O} \end{bmatrix}$$

The general solution is of the form

$$\dot{W}(t) = \exp(At)W(0) \qquad (3.29)$$

where both A and $\exp(At)$ in this case are diagonalizable . Now one can find that the eigenvalues of the matrix A are the square roots of eigenvalues of the matrix A_1,which

are given by

$$\lambda_i(A_1) = \frac{-4}{h^2}\sin^2((2i-1)\pi/4N) \quad , i = 1(1)N \qquad (3.30)$$

Since $\lambda_i(A_1)$ are all real and negative, then the eigenvalues of A are all pure imagin- ary. Since the solution depends on the exponential of eigenvalues whose real parts are equal to zero, then a numerical solution of equation (3.28) is marginal stable.

So far, we considered the stability analysis of the method of lines for parabolic and hyperbolic Partial differential equations. As a result of such analysis the resulting system is of the initial value type in ordinary differential equations, this because the discretization takes place in the x-direction. It should also be mentioned that, when the discretization takes places in the time direction, the resulting system will be of the boundary value type in ordinary differential equations. As far as the elliptic equation is concerned (see section (4))the resulting system will also be of boundary value type.

4. STABILITY ANALYSIS FOR ELLIPTIC EQUATIONS

In view of the too complicated behavior of the boundary value problems than that of the Initial value one , it is not surprising that no serious attempts except that of Jones et al [12]have been made to analyze the stability of the

method of lines for elliptic Partial differential equations, and even for the Parabolic and hyperbolic equations with discretization in the time direction.
Thus, a study must now be developed to fill the gap left for the stability analysis of the method of lines for;

(1) The elliptic Partial differential equations.

(2) The parabolic and hyperbolic Partial differential equations under discretization in the time direction.

In this section technique for the stability analysis of elliptic partial differential equations when treated by the method of lines will be established.

In order to analyze the stability of the method of line for elliptic partial differential equations , we consider three examples , which are the Poisson equation in two dimensions , Biharmonic equation, and a nonlinear second order elliptic equation .

4.1 The Poisson's Equation :

We consider the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \quad ,(x,y)\in D \qquad (4.1)$$

where $D=\{(x,y):0<x<a \ , \ 0<y<b\}$ and $f(x,y)\in C(D)$. Subject to the boundary conditions

$$u(0,y)=0=\frac{\partial u(a,y)}{\partial x} \qquad (4.2)$$

and

$$u(x,0)=0 \quad , u(x,b)=\sin\frac{\pi x}{a} \qquad (4.3)$$

When applying method of lines , with discretization in y direction , we obtain using "3" point scheme

$$\frac{d^2u}{dx^2}+\frac{(u_{i+1}-2u_i+u_{i-1})}{h^2}=f(x,y)+E_T \quad ,i=1(1)N \qquad (4.4)$$

Where $h=\dfrac{b}{(N+1)}$, and E_T is of $O(h^2)$ (we drop E_T from now on) ; and subject to

$$u_i(0)=0=\frac{\partial u_i(a)}{\partial x} \qquad (4.5.a)$$

$$u_0(x) = 0 \quad , u_{N+1}(x) = \sin\frac{\pi x}{a} \tag{4.5b}$$

Now let

$$\frac{du_i}{dx} = v_i \quad , \quad i = 1(1)N \tag{4.6}$$

then

$$\frac{dv_i}{dx} = \frac{2u_i - (u_{i+1} + u_{i-1})}{h^2} + f_i(x) \quad , i = 1(1)N \tag{4.7}$$

Hence , we can write the system , in the matrix form as

$$\frac{d}{dx}\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_N \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} = \left[\begin{array}{cccccc} & & & \cdot & & \\ & & & \cdot & & \\ & O_{N,N} & & \cdot & & I_N \\ & & & \cdot & & \\ \hline \frac{2}{h^2} & \frac{-1}{h^2} & 0 \; \ldots & 0. & & \\ \frac{-1}{h^2} & \frac{2}{h^2} & \frac{-1}{h^2} & 0. & & \\ & \ldots & & \cdot & O_{N,N} & \\ 0 & \ldots & 0 & \frac{-1}{h^2} \; \frac{2}{h^2}. & & \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_N \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_N(x) - \sin\frac{\pi x}{a} / h^2 \end{bmatrix} \tag{4.8}$$

with the conditions

$$u_i(0) = 0 \quad , i = 1(1)N \tag{4.9a}$$

$$v_i(a) = 0 \quad , \quad i = 1(1)N \tag{4.9b}$$

The system (4.8) with (4.9) is a first order boundary value problem in ordinary differential equations . To solve such system we must find the value of v_i , $i = 1(1)N$, at $x = 0$, which is not known . Fortunately a transformation from boundary value problem to initial value problem is possible by using one of the shooting methods [13,19] . One of the most powerful method of shooting is the adjoints method .

Rewrite equation (4.8) in a compact form as

$$\dot{W} = A\,W + \underline{F} \qquad (4.10)$$

where

$$W^T = [U^T, V^T],\ U^T = [u_1\ u_2\ \dots\ u_N],\ V^T = [v_1\ v_2\ \dots\ v_N]$$

and $A = \lfloor a_{ij} \rfloor$ is a matrix of order $2N \times 2N$ where

$$a_{ij} = \begin{cases} 0 & , i,j = 1(1)N; j = (N+1)(1)2N \\ \delta_{ij} & , i = 1(1)N\ , j = (N+1)(1)2N \\ \dfrac{2}{h^2} & , i = j\ , j = (N+1)(1)2N \\ \dfrac{-1}{h^2} & , |i-j| = 1\ , i = (N+1)(1)2N \\ 0 & , |i-j| \geq 2\ , i = (N+1)(1)2N \end{cases} \qquad (4.11)$$

and $\underline{F}^T = \lfloor \underline{O}\ ,\ \underline{f}^T \rfloor$, $\underline{f}^T = \lfloor f_1, f_2, \dots, f_N - \sin\frac{\pi x}{a} / h^2 \rfloor$

Using the adjoint method , with adjoint system

$$\dot{Z}(x) = A^T\ Z(x) \qquad (4.12)$$

Then the identity of method is

$$\sum_{i=1}^{2N} Z_i^{(m)}(a)W_i(a) - \sum_{i=1}^{2N} Z_i^{(m)}(0)W_i(0) = \int_0^a \sum_{i=1}^{2N} Z_i^{(m)}(x)F_i(x)\ dx \quad m = 1(1)N \qquad (4.13)$$

with a suitable terminal conditions for the adjoint system

$$Z_i^{(m)}(a) = \begin{bmatrix} 1 & , & i = i_m \\ 0 & , & i \neq i_m \end{bmatrix} \qquad m = 1(1)N \qquad (4.14)$$

where $i_m = (N+1)(1)2N$ (for our problem)

Now equation (4.13) can be rewritten as follows

$$\begin{bmatrix} Z_{N+1}^{(1)}(0) & Z_{N+2}^{(1)}(0) & \dots & Z_{2N}^{(1)}(0) \\ Z_{N+1}^{(2)}(0) & Z_{N+2}^{(2)}(0) & \dots & Z_{2N}^{(2)}(0) \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ Z_{N+1}^{(N)}(0) & Z_{N+2}^{(N)}(0) & \dots & Z_{2N}^{(N)}(0) \end{bmatrix} \begin{bmatrix} W_{N+1}(0) \\ W_{N+2}(0) \\ \cdot \\ \cdot \\ \cdot \\ W_{2N}(0) \end{bmatrix} = \begin{bmatrix} W_{N+1}(a) - \sum_{i=1}^{N} Z_i^{(1)}(0)W_i(0) - \int_0^a \sum_{i=1}^{2N} Z_i^{(1)}F_i dx \\ W_{N+2}(a) - \sum_{i=1}^{N} Z_i^{(2)}(0)W_i(0) - \int_0^a \sum_{i=1}^{2N} Z_i^{(2)}F_i dx \\ \cdot \\ \cdot \\ \cdot \\ W_{2N}(a) - \sum_{i=1}^{N} Z_i^{(N)}(0)W_i(0) - \int_0^a \sum_{i=1}^{2N} Z_i^{(N)}F_i dx \end{bmatrix} \qquad (4.15)$$

Using terminal conditions (4.14) and the boundary conditions (4.9a) and (4.9b) one can rewrite (4.15) as follows

$$\begin{bmatrix} Z^{(1)}_{N+1}(0) & Z^{(1)}_{N+2}(0) & \cdots & Z^{(1)}_{2N}(0) \\ Z^{(2)}_{N+1}(0) & Z^{(2)}_{N+2}(0) & \cdots & Z^{(2)}_{2N}(0) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ Z^{(N)}_{N+1}(0) & Z^{(N)}_{N+2}(0) & \cdots & Z^{(N)}_{2N}(0) \end{bmatrix} \begin{bmatrix} W_{N+1}(0) \\ W_{N+2}(0) \\ \cdot \\ \cdot \\ \cdot \\ W_{2N}(0) \end{bmatrix} = \begin{bmatrix} -\int_0^a \sum_{i=1}^{2N} Z_i^{(1)} F_i dx \\ -\int_0^a \sum_{i=1}^{2N} Z_i^{(2)} F_i dx \\ \cdot \\ \cdot \\ \cdot \\ -\int_0^a \sum_{i=1}^{2N} Z_i^{(N)} F_i dx \end{bmatrix} \qquad (4.16)$$

To solve the system (4.16) for the missing initial conditions $W_i(0)$,$i = N+1(1)2N$,

we must have the values of $Z_i^{(m)}(x)$, $i = 1(1)2N$, $m = 1(1)N$. To do this we integrate the adjoint system (4.12) backward from a to zero, with the terminal conditions (4.14) with storing the profiles.On the other hand one can solve the adjoint system(4.12) with (4.14) analytically which is the way to be considered in this work .

The solution of the adjoint system , depend on changing the variable x by

$$t = a - x \qquad (4.17)$$

And whence the adjoint system (4.12) becomes

$$\dot{Z}^{(m)}(t) = A^T \; Z^{(m)}(t) \qquad , m = 1(1)N \qquad (4.18)$$

The general solution of (4.18) is

$$\dot{Z}^{(m)}(t) = \exp(A^T t) \; Z^{(m)}(0) \qquad , m = 1(1)N \qquad (4.19)$$

Now the problem is to find $\exp(A^T t)$ which in turns depends on the evaluation of the eigenvalues of the matrix A and some related matrices .

The eigenvalues of the matrix A are given as

$$\lambda_i = \frac{2}{h}\sin\frac{i\pi}{2(N+1)} \qquad , \quad i = 1(1)N$$

$$\lambda_{i+N} = \frac{-2}{h}\sin\frac{i\pi}{2(N+1)} \qquad , \quad i = 1(1)N \qquad (4.20)$$

Due to the positvety of the eigenvalues of A (equations(4.20))to gather with. the round off error (of the numerical computation) is one of the heavy sources of the instability of the solution of the system (4.18).(This will be considered in details at the end of this section). In order to complete our analysis it remains to consider the effect of the number of lines on the missing initial conditions as another source of instability of the solution. To do so, let us first consider the eigenvectors of the matrix A which are

$$\underline{x}^i = \begin{bmatrix} \underline{x}_1^i \\ \lambda_i \underline{x}_1^i \end{bmatrix} \quad , \quad i = 1(1)N \qquad (4.21a)$$

and

$$\underline{x}^{i+N} = \begin{bmatrix} \underline{x}_1^i \\ \lambda_{i+N} \underline{x}_1^i \end{bmatrix} \quad , \quad i = 1(1)N \qquad (4.21b)$$

Secondly, we estimate the upper and lower limits of the eigenvalues of the matrix A

$$|\lambda|_i = \left| \frac{2}{h} \sin \frac{i\pi}{2(N+1)} \right| \le \frac{2}{h} \quad , \quad i = 1(1)N$$

$$|\lambda_{i+N}| = \left| \frac{-2}{h} \sin \frac{i\pi}{2(N+1)} \right| \ge \frac{-2}{h} \quad , \quad i = 1(1)N \qquad (4.22)$$

one can use these estimates directly to study the worst case of the system (4.18) , which implies the maximization of the difference of eigenvalues (stiff ratio) . By taking equation(4.21)into consideration , one may approximate the $\exp(A)$ by

$$\exp(A) = T \left[\begin{array}{c:c} e^{\frac{h}{2}} \mathrm{I}_N & \mathrm{O}_{N,N} \\ \hdashline \mathrm{O}_{N,N} & e^{\frac{-h}{2}} \mathrm{I}_N \end{array} \right] T^{-1} \qquad (4.23)$$

where T is the matrix of eigenvectors of A , and takes the form

$$T = \begin{bmatrix} X & X \\ X\,D & -X\,D \end{bmatrix} \qquad (4.24)$$

where X is the matrix of eigenvectors of A_1 , which is orthogonal one , since A_1 is a symmetric matrix , then D is the diagonal matrix with $D_{ii} = \frac{2}{h} \; , i = 1(1)N$.

By using the orthogonality property of matrix X ,one can put the T^{-1} matrix in the following form

$$T^{-1} = \begin{bmatrix} \frac{1}{2}X^T & \frac{1}{2}D^{-1}X^T \\ \frac{1}{2}X^T & -\frac{1}{2}D^{-1}X^T \end{bmatrix} \tag{4.25}$$

substitute (4.23) , and (4.24) in equation (4.22) , to obtain

$$\exp(A) = \begin{bmatrix} \cosh\frac{2}{h}\mathbf{I}_N & \frac{h}{2}\sinh\frac{2}{h}\mathbf{I}_N \\ \frac{2}{h}\sinh\frac{2}{h}\mathbf{I}_N & \cosh\frac{2}{h}\mathbf{I}_N \end{bmatrix} \tag{4.26}$$

Then the $Z^{(m)}$ becomes

$$Z^{(m)}(t) = \begin{bmatrix} \cosh\frac{2}{h}\mathbf{I}_N & \frac{2}{h}\sinh\frac{2}{h}\mathbf{I}_N \\ \frac{h}{2}\sinh\frac{2}{h}\mathbf{I}_N & \cosh\frac{2}{h}\mathbf{I}_N \end{bmatrix} Z^{(m)}(0) \tag{4.27}$$

since

$$Z^{(m)}(t=0) = Z^{(m)}(t=a) = e_{m+N} \qquad , \quad m = 1(1)N, \tag{4.28}$$

Hence

$$Z(t) = \begin{bmatrix} \frac{2}{h}\sinh\left(\frac{2}{h}t\right)\mathbf{I}_N \\ \cosh\left(\frac{2}{h}t\right)\mathbf{I}_N \end{bmatrix} \tag{4.29}$$

Now the system (4.16) is reduced to

$$\cosh\left(\frac{2}{h}a\right)\begin{bmatrix} W_{N+1}(0) \\ W_{N+2}(0) \\ \cdot \\ \cdot \\ \cdot \\ W_{2N}(0) \end{bmatrix} = \begin{bmatrix} -\int_0^a \cosh\left(\frac{2}{h}x\right)f_i dx \\ -\int_0^a \cosh\left(\frac{2}{h}x\right)f_2 dx \\ \cdot \\ \cdot \\ \cdot \\ -\int_0^a \cosh\left(\frac{2}{h}x\right)\left[f_N - \sin\frac{\pi x}{a}/h^2\right]dx \end{bmatrix} \tag{4.30}$$

By using the mean value theorem for integration , system (4.29) is reduced to

$$\begin{bmatrix} W_{N+1}(0) \\ W_{N+2}(0) \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ W_{2N}(0) \end{bmatrix} = \frac{-1}{\cosh(\frac{2}{h}a)} \begin{bmatrix} -f_1(\xi_1)\int_0^a \cosh(\frac{2}{h}x)dx \\ -f_2(\xi_2)\int_0^a \cosh(\frac{2}{h}x)dx \\ \cdot \\ \cdot \\ \cdot \\ -[f_N(\xi_N)-\sin\frac{\pi\bar{\xi}_N}{a}/h^2]\int_0^a \cosh(\frac{2}{h}x)dx \end{bmatrix} \tag{4.31}$$

where

$$\xi_i \;,\bar{\xi}_N \;\in(0,a) \;,i=1(1)N$$

Let us put

$$M = Max(Max(f_i(\xi_i),(f_N(\xi_N)-\sin(\tfrac{\pi\bar{\xi}_N}{a})/h^2)) \;,$$
$$0<\xi_i \quad , \quad \bar{\xi}_N < a \quad ,i=1(1)N$$

Hence , we can rewrite (4. 31) as follows

$$\begin{bmatrix} W_{N+1}(0) \\ W_{N+2}(0) \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ W_{2N}(0) \end{bmatrix} = \frac{-M\,h}{2}\tanh(\frac{2}{h}a) \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} \tag{4.32}$$

Now , all the initial conditions of the original system (4.10) are to be found , and the general solution is

$$W(x)=\exp(A\,x)\left[W(0)+\int_0^x \exp(-Ax_1)\underline{F}(x_1)\,dx_1\right] \quad ,x\in(0,a] \tag{4.33}$$

where

$$W_i(0)=0 \quad ,1(1)N \tag{4.34}$$

and

$$W_{i+N}(0)=\frac{-Mb}{2(N+1)}\tanh\frac{2(N+1)i}{b} \quad , \; i=1(1)N \qquad (4.35)$$

substitute (4.25) , (4.34) and (4.35) in equation (4.33), to obtain

$$W_i(x)=\frac{Mb^2}{4(N+1)^2}\left[\cosh\frac{2(N+1)}{b}(-\tanh\frac{2(N+1)}{b}a\sinh\frac{2(N+1)}{b}x-1\right], \; i=1(1)N$$

$$W_i(x)=\frac{Mb}{2(N+1)}\sinh\frac{2(N+1)}{b}x \quad , \; i=1(1)N \qquad (4.36)$$

There are two different reasons of instability:

(1) The instability, due to the positive exponential growth of the solution with the number of lines, increases .

(2) The inherent instability of the resulting solution of the system of differential equations due to the high stiff ratio between eigenvalues ,and it becomes severe when the number of lines increases.

It is noticeable that, the above two sources of the instability, are deduced from the nature of the analytical solution as it stands in equation (4. 60). But when the given equation (4.1) is treated numerically by the method of lines, the two common errors for the numerical computation, namely the truncation and inherent errors will be magnified by the above sources of instability.

Now we are about to deal with the minimization of the effects of instability. For instance, since it is the product (N+1) which causes instability, it may in some cases be worthwhile using smaller integration distances x . This leads to the idea of the multi shooting technique which means that the interval of integration is divided into multi subintervals and some smoothing criterion is satisfied at the shooting points which in the interior of the subintervals. Also using a higher order difference scheme (say, "5" point scheme) for the second derivative enables us to use fewer lines to achieve the same accuracy for this second derivative. By using smaller number of lines we reduce the exponential growth of errors in the solution. Also the stiff ratio is reduced considerably. To avoid the instability due to the second effect mentioned above, we use an integration routine of the differential equation, which deal, with a stiff system.

4.2 The Non homogenous Biharmonic Equation

Now we consider the stability analysis for the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=f(x,y)\,,\;(x,y)\in D \qquad (4.37)$$

where $D = \{(x,y): 0 \le x \le a \,,\, 0 \le y \le b\}$ and $f(x,y) \in C(D)$ with the boundary conditions

$$u(0,y) = \frac{\partial^2 u(0,y)}{\partial x^2} = 0 = u(a,y) = \frac{\partial^2 u(a,y)}{\partial x^2}$$
$$u(x,0) = \frac{\partial^2 u(x,0)}{\partial y^2} = 0 = u(x,b) = \frac{\partial^2 u(x,b)}{\partial y^2} \qquad (4.38)$$

We consider two poisson equations ,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = w(x,y) \ ,$$
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y) \qquad (4.39)$$

instead of the equation (4.37) . Now , as mentioned for Poisson's equation , the two partial differential equations are transformed to the following two coupled system of ordinary differential equations

$$\frac{d^2 u_i}{dx^2} + \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = w_i(x) \quad ,u_i(0) = u_i(a) = 0 \ , \ i = 1(1)N \qquad (4.40)$$

and

$$\frac{d^2 w_i}{dx^2} + \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} = f_i(x) \quad ,w_i(0) = w_i(a) = 0 \ , \ i = 1(1)N \qquad (4.41)$$

where $h = \dfrac{b}{(N+1)}$.

Let

$$\frac{du_i}{dx} = v_i \quad , \ i = 1(1)N \qquad (4.42)$$

and

$$\frac{dw_i}{dx} = z_i \quad , \ i = 1(1)N \qquad (4.43)$$

Hence the coupled system (4.40) and (4.41) , can be rewritten in the matrix vector form as follows

$$\frac{d}{dx}\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_N \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} = \left[\begin{array}{c:c} O_{N,N} & I_N \\ \hdashline A_1 & O_{N,N} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_N \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_N \end{bmatrix} \qquad (4.44)$$

and

$$\frac{d}{dx}\begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_N \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_N \end{bmatrix} = \left[\begin{array}{c:c} O_{N,N} & I_N \\ \hdashline A_1 & O_{N,N} \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_N \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{bmatrix} \qquad (4.45)$$

with

$u_i(0) = u_i(a) = 0 \ , \ i = 1(1)N$ (4.46)

and

$w_i(0) = w_i(a) = 0 \ , \ i = 1(1)N$ (4.47)

where the matrix A_1 as it is defined for Poisson's equation of the previous analysis . The system (4.44) is coupled with the system (4.45) . Hence we solve system (4.44) with boundary conditions (4.46) , then solve the system (4.45) with boundary conditions (4.47).

To solve the system (4.45) with (4.47) , we define firstly its adjoint system as

$$\dot{\underline{Q}}^{(m)}(x) = -A^T \ \underline{Q}^{(m)}(x) \ , \ m = 1(1)N \qquad (4.48)$$

with terminal conditions

$\underline{Q}^{(m)}(a) = \underline{e}_m \quad ,m = 1(1)N$ (4.49) one can obtain the solution of (4.48) by following the same analysis as in deriving equation (4.18) in the form

$$\underline{Q}(t)=\begin{bmatrix}\cosh(\frac{2}{h}t)\,\mathbf{I}_N\\ \frac{h}{2}\sinh\,(\frac{2}{h}t)\,\mathbf{I}_N\end{bmatrix} \tag{4.50}$$

where $t=a-x$, then the initial value to Q takes the form

$$\underline{Q}(t=a)=\underline{Q}(x=0)=\begin{bmatrix}\cosh(\frac{2}{h}a)\,\mathbf{I}_N\\ \frac{h}{2}\sinh(\frac{2}{h}a)\,\mathbf{I}_N\end{bmatrix} \tag{4.51}$$

we use equation (4.51) in identity of adjoint method , one can obtain :

$$\frac{h}{2}\sinh\,(\frac{2}{h}a)\begin{bmatrix}Z_1(0)\\ Z_2(0)\\ \cdot\\ \cdot\\ \cdot\\ Z_N(0)\end{bmatrix}=-\int_0^a\begin{bmatrix}\frac{h}{2}\sinh\,(\frac{2}{h}x)f_1(x)\,dx\\ \frac{h}{2}\sinh\,(\frac{2}{h}x)f_2(x)\,dx\\ \cdot\\ \cdot\\ \cdot\\ \frac{h}{2}\sinh\,(\frac{2}{h}x)(f_N(x)-\sin\frac{\pi x}{a}/h^2)\,dx\end{bmatrix} \tag{4.52}$$

Then , the missing initial conditions are

$$\begin{bmatrix}Z_1(0)\\ Z_2(0)\\ \cdot\\ \cdot\\ \cdot\\ Z_N(0)\end{bmatrix}=-\frac{h}{2}M\frac{\cosh\frac{2}{h}a-1}{\sin sh\frac{2}{h}a}\begin{bmatrix}1\\ 1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{bmatrix} \tag{4.53}$$

where

$$M=\max\left(\max\left|f_i\;,\;\xi_i\right|\;,\;f_N(\xi_N)-\sin\tfrac{\pi\bar{\xi}}{a}N/h^2)\right),i=1(1)N$$

$$0<\xi_i\;,\bar{\xi}_i<a \qquad\qquad <i=1(1)N$$

The solution to the original problem is now obtainable , as

$$W_i(x)=\frac{-Mb^2}{4(N+1)^2}\left[1-\cosh\frac{2b}{N+1}x-\sinh\frac{2b}{N+1}x\frac{\cosh\frac{2b}{N+1}a-1}{\sin sh\frac{2b}{N+1}a}\right],i=1(1)N \quad (4.54)$$

and

$$Z_i(x)=\frac{-Mb}{2(N+1)}\left[-\sinh\frac{2b}{N+1}+\cosh\frac{2b}{N+1}\times\frac{\cosh\frac{2b}{N+1}a-1}{\sin sh\frac{2b}{N+1}a}\right],i=1(1)N \quad (4.55)$$

It is easy to show that the missing initial conditions for equation (4.44) $v_i(0)\quad ,i=1(1)N$ are

$$v_i(0)=\frac{3Mh^2}{32}\left(\cosh\frac{2}{h}a-\cosh^2\frac{2}{h}a\right)+\frac{Mh^3}{8}a\frac{\cosh\frac{2}{h}a-1}{\sin sh\frac{2}{h}a}\ ,\ i=1(1)N \qquad (4.56)$$

Then the solution of equation (4.44) is be represented as

$$u_i(x)=l(x)\cosh\frac{2}{h}x+\frac{h}{2}\sin sh\frac{2}{h}x(v_i(0)+m(x))\ ,\ i=1(1)N \qquad (4.57)$$

and

$$v_i(x)=\frac{2}{h}\,l(x)\sin sh\frac{2}{h}x+\cosh\frac{2}{h}x(v_i(0)+m(x))\ ,\ i=1(1)N \qquad (4.58)$$

where

$$l(x)=\frac{Mb^3}{8}\left[\frac{h}{2}\left(\cosh\frac{2}{h}x-\cosh^2\frac{2}{h}x-\tfrac{1}{4}\frac{\cosh\frac{2}{h}a-1}{\sinh\frac{2}{h}a}\sinh\frac{4}{h}x\right)+\frac{x}{2}\frac{\cosh\frac{2}{h}a-1}{\sinh\frac{2}{h}a}\right]$$

$$,i=1(1)N \qquad (4.59)$$

and

$$m(x)=\frac{-Mb^2}{4}\left[\frac{h}{2}\left(\sinh\frac{2}{h}x-\tfrac{1}{4}\sinh\frac{4}{h}x-\frac{\cosh\frac{2}{h}a-1}{\sinh\frac{2}{h}a}\sinh^2\frac{2}{h}x\right)-\frac{x}{2}\right],i=1(1)N \quad (4.60)$$

It is worthy at the present to conclude the following notes:

The source of the instability for Poisson's equation is attributed to (as mentioned previously) the factor $2(N+1)x$ due to its positively and its dependence on the number of lines N . A result which shows how the instability of the solution is affected by the number of lines. While for the biharmonic equation the source is due to the factor $4(N+1)x$,which shows the linearity of the error growth with respect to N as the Poisson's equation but with double rate.This is a new fact in contradiction to what is usually expected of quadratic dependence of the error growth with respect to N .

4.3 Nonlinear Second order Elliptic Equation :

Now, to illustrate the generality of the considered method, let us consider the following nonlinear equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\exp(u)\,,\; 0\leq(x,y)\leq 1 \qquad (4.61)$$

with homogeneous boundary conditions . We apply the method of lines to equation (4..61) , to get

$$\frac{du_i}{dx}=v_i \qquad ,\; i=1(1)N \qquad (4.62)$$

$$\frac{dv_i}{dx}=\frac{-1}{h^2}\left(u_{i+1}-2u_i+u_{i-1}\right)+\exp(u_i) \qquad ,i=1(1)N \qquad (4.63)$$

Such a system needs an iterative process to obtain its solution.We use the quasiline- arization technique to transform the system (4.62) to linear system and then the

solution can be obtained iteratively . In the n th stage of iteration the system becomes

$$\frac{du^{(n)}{}_i}{dx}=v^{(n)}{}_i \qquad ,\; i=1(1)N, n=1(1)... \qquad (4.64)$$

$$\frac{dv^{(n)}{}_i}{dx}=\frac{-1}{h^2}\left(u^{(n)}{}_{i+1}-2u^{(n)}{}_i+u^{(n)}{}_{i-1}\right)+(\exp(u_i^{(n-1)})+(u^{(n)}{}_i-u_i^{(n-1)})\exp(u_i^{(n-1)}) \qquad ,i=1(1)N \qquad (4.65)$$

For the theory of the quasilinearization ,one can be consulted the references[2,19] .We consider only the solution ,when n is equal 1 and take the zero solution as an initial gauss . In general $\exp(u_i^{(n)})$ can be written as follows

$$\exp(u_i^{(n)}) = (1 + u_i^{(1)}) \prod_{j=1}^{n-1} (1 + u_i^{(j+1)} - u_i^{(j)}) \qquad (4.66)$$

for $n = 1$, the $\exp(u_i^{(n)})$ is

$$\exp(u_i^{(1)}) = (1 + u_i^{(1)}) \qquad (4.67)$$

Now , we rewrite the system in a compact form for $n = 1$ as follows

$$\begin{bmatrix} \underline{u}' \\ \underline{\dot{v}} \end{bmatrix} = \begin{bmatrix} \mathrm{O}_{N,N} & \cdot & \mathrm{I}_N \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ A'_1 & \cdot & \mathrm{O}_{N,N} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (4.68)$$

where $A'_1 = A_1 + \mathrm{I}$. The boundary conditions for system (4.68) are

$$\underline{u}(0) = \underline{0}$$
$$\underline{v}(1) = \underline{0} \qquad (4.69)$$

This system cannot be solved , directly , thus we first estimate the missing conditions $\underline{v}(0)$.

The eigenvalues of the system (4.68) approximately given by :

$$\lambda_i = \frac{2}{h}\sin\frac{i\pi}{2(N+1)}\left[1 + \tfrac{1}{8}\frac{h^2}{\sin^2(i\pi/2(N+1))}\right],$$
$$\lambda_{i+N} = -\lambda_i \ , \ i = 1(1)N \qquad (4.70)$$

The missing initial conditions are approximately given by

$$v_i^{(1)}(0) \cong -\frac{h}{2}\tanh(\frac{2}{h}) \qquad (4.71)$$

Then the solution of (4.68) is

$$u_i^{(1)}(x) = \frac{1}{4(N+1)^2}\left[\cosh 2(N+1)x - \tanh 2(N+1)\sinh 2(N+1x-1\right],$$

$$v_i^{(1)}(x) = \frac{N+1}{2}\left[\sinh 2(N+1)x\right] \ , i = 1(1)N \qquad (4.72)$$

Then one can use this technique to construct the solution of stage 2 , and so on .

Despite that the original equation is nonlinear its solution (4.72) (first iteration) is typical to that Poisson equation . Consequently its stability condition is exactly the same as that of Poisson's equation . Moreover , due to the linearity of error growth with respect to the number of lines , one can conclude an important result that the invariance of the stability conditions with respect to iterative computational algorithm .

5. A GENERAL ALGORITHM FOR THE STABILITY ANALYSIS.

The stability of the method of lines for partial differential equations represents the most important factor for their solutions and at the same time is a critical factor that should be handled carefully. The importance lies in its unique ability of judging acceptable solution for the given equation , of being critical is due to its dependence on the nature of the eigenvalues of the matrix representation connection with their number . A typical example of the danger inherent from such dependence is the application of the method of lines to partial differential equation of elliptic type , for which there exist exponential dependence of the eigenvalues of the matrix representation and the number of lines of the spatial variable , which in turn may lead to exponential growth of errors as the number of lines increases .

On the other hand , the parabolic and hyperbolic partial differential equations when treated numerically by method of lines gives non positive exponential dependence of the eigenvalues of the matrix representation on the number of lines , which leads to exponential decay of the errors as the number of lines increases . This last fact may explain why the stability analysis of the method of lines for parabolic and hyperbolic equations are of common appearance in various publications . As for as the computational design of the stability analysis of method of lines is concerned , any plan for a general algorithm should consider all types of partial differential equations without particularization to certain type . This is because , in mathematical physics problems and other branches , all types appear with equal chance , moreover in some problem of mixed type we have to consider more than one type at the same time . Consequently , the need is ergent for such general algorithm , in what follows , the computational steps for such algorithm one given just to illustrate the rate the strategy of our plane for package formation for method of lines

1. Discretize all independent variables , but one .
2. All partial derivatives of the dependent variable are approximated by suitable finite difference operators .
3. If the resulting system of ordinary differential equations is of the boundary value type .
   <u>Then</u> go to step 4 .
   <u>Else</u> go to step 5.
4. Transform the boundary value problem into initial value problem

using the adjoint shooting technique or other suitable technique .

5. The resulting system is of the form

(A) $B(x)\,\underline{\dot{W}}(x) = A(x)\ \underline{W}(x) + \underline{F}(x)$

For linear partial differential equations where $\underline{W}(x)$ is the solution vector .

(B) $D(x)\,\underline{\dot{W}}^{(n)}(x) = C(x)\ \underline{W^{(n)}}(x) + G(x, W^{(n-1)}(x), W^{(n-1)}(x))$

For nonlinear partial differential equations when using quasilinearization technique to transform the nonlinear system to sequence of linear system which is solved iteratively where (n) is the iteration stages , $n = 1(1)...$.

6 . If The real part of the eigenvalues of the matrix representation are nonpositive

Then The solution is stable .

Else The solution is unstable , so using the previous recommendation to minimize the effect of instability .

REFERENCES

[1] Ames,W.F. Numerical Methods for Partial differential equations,2nd ED.(Academic Press. New York and San Francisco) (1977).

[2] Bellman, R. and kalaba, R. Quasilinearization and Nonlinear Boundary Value Problems (American Elsevier, New York) (1965).

[3]Chin-Yuan Lin , " The method of Lines for Solving a Class of Abstract differential equations " , Nonlinear Analysis 45, pp. 383-390, (2001).

[4] Gerisch,A. and J. G. Verwer ," Operator Splitting and Approximate Factorization for Taxis-Diffusion Reaction Models" , Applied Numerical Mathematics,Vol.42,pp.159-176(2002).

[5] Fisher,M.," Higher Order Differences in the Analogue Solution of Partial Differential Equations " , J. ACM , Vol.3,pp.325-347(1955) .

[6] Forsythe,G.E. and Wasow,W.R." Finite Difference Method for Partial Differential equations John Wiley, New York ,(1960).

[7] Gustafsson , B. , "On the Implementation of Boundary Conditions for the Method of Lines" , BIT , Vol.38,No.2,pp.293-314(1998).

[8] Heydweiller,J.C. and R. F. Sincovec," A stable Difference Scheme for the Solution of Hyperbolic Equations Using the method of lines " , J. comp.Phys.,Vol.22 , pp.377-388(1976) .

[9] Hicks,J.S. and J. Wei., " Numerical Solution of Parabolic Partial Differential Equations with Two-Point Boundary Conditions by Use of the Method of Lines " , J. ACM ,Vol.14, No.3 ,pp.549-562(1967) .

[10] Hindmarsh, A.C." Gear : Ordinary Differential Equation System Solver " ,Lawrence Livemore Laboratory , Report UCID- 30001 , Rev. ,3(1974).

[11] Jolley, L.B.W." Summation of Series "(2 nd Rev. Ed.), DoverPublications , New York (1976) .

[12] Jones,D. J. , J.C.,South and E.B Klunker , " On the Numerical Solution of Elliptic Partial Differential Equations by the Method of Lines" , J. of Computational physics,Vol.9 ,pp.496-527(1972) .

[13] Keller,H.B."Numerical Methods for Two Point Boundary Value Problems" ,9Ginn- Blaisdell , Walthan. Mass). (1968) .

[14] Kopal,Z."Numerical Analysis " , Wiley , New York , (1955) .

[15] Kreiss,H. and G. Scherer , "Method of Lines for Hyperbolic Differential Equations",SIAM J. Numer. Anal. , Vpl.29 , No.3,pp.640-646(1992) .

[16]Miller, K.S. " Linear Differential Equations " , (Routiedge and Kegan , london) , (1963) .

[17] Reddy,S. C. and L. N. Trefethen," Stability of the Method of Lines ", Numer. Math. Vol. 62,pp.235-267(1992) .

[18] Richtmyer,R.D. and Morton, K.W. Difference Method for initial Value Problems (Wiley. New York) (1967).

[19]Roberts,S.M. and Shipman,J.S." Two Point Boundary Value Problems Shooting Methods " , (American Elsevier, New York) , (1972) .

[20] Shampine,L. and Gordon,M." Computer Solution of Ordinary Differential Equations " , (Freeman, San Francisco) , (1975) .

[21] Smith, G.D. ,"Numerical Solution of Partial Differential Equations ", 2nd ED. Clarendon Press. Oxford , (1978).

[22] Strikwerda,J.C. , J.of comp. Phys. 34(1980) .

[23] Thomee,V. , SIAM J. Numer. Anal. ,4(1967) .

[24]Watanabe,D. and Flood,J. , Math. Comp. 28(1974) .

[25] Zafarullah,A.," Application of the Method of Lines to Parabolic PartialDifferential Equations With Error Estimates " , Journal of theAssociation for Computing Machinery , Vol.17,No.2, pp.294-302(1970).