Solution of Parabolic Navier-Stokes Equations for The Entrance Region-Flow between Two Parallel Plates

حل معادلات نافير ستوكس المكافئة للسريان بين لوحين متوازيين في منطقة الدخول

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خلاصة: في هذا البحث تم نظريا فحص نمو السريان بين لوحين متوازيين في منطقة الدخول. وتم وصف السريان بواسطة معادلات نافير ستوكس المكافئة، ولطبيعة هذه المعادلات فانها يمكن حلها بتطبيق طريقة الحل المحلي التشابهي. طبقا لهذه الطريقة فان الصورة اللابعدية لمعادلات كمية الحركة يمكن تحويلها الى معادلات تفاضلية عادية. هذه الصورة المعدلة لمعادلات الحركة يمكن حلها عدديا بطريقة رونج كوتا للمعادلات التفاضلية العادية مصحوبة بطريقة الرصد لمسائل القيم الحدية. في هذا العمل تم حساب معامل الاحتكاك في اماكن مختلفة بطول الممر لممرات ذات ارتفاعات مختلفة. كذلك عن طريق هذا الحل يمكن معرفة توزيع السرعات عند اى مقطع للممر. في هذا البحث درست ثلاث ممرات ذات ارتفاعات طبقا لقيم رقم رينولدز ١٠٠، ٢٠٠، ٣٠٠ .

Abstract- The development of the flow field in the entrance region is theoretically examined, for the case of flow between two parallel flat plates. The flow field is described by the parabolic Navier-Stokes equations. Because of the nature of these equations; it is convenient to solve them by the application of the local similarity solution method. According to this method, dimensionless form of momentum equations are transferred to ordinary differential equations. These two modified equations are solved numerically by Runge-Kutta method of ordinary differential equations accompanied with shooting method of boundary value problems.

The values of coefficient of friction are calculated at different positions along the passage. Also the velocity profile at any position, according to this approach, can be obtained. Three passages, in this work, are studied; with Reynolds number (based on the half of the height of the passage) of 100,200 and 300.

1. INTRODUCTION

A complete knowledge of the mechanism of the flow of fluids in pipes and channels is basic to the understanding of heat transfer processes. The developing of velocity profile in the entrance region of ducts is of a great importance in case of combined entrance region laminar forced convection.

As surveyed by Kakac and Yener [2], different methods have been developed to solve the problem of laminar forced convection in combined entrance region of a duct. These methods are based on the hydrodynamic boundary layer approximation. Later, Wasel [4] made a local similarity solution of laminar forced convection in entrance region for flow between two parallel plates. According to this solution the pressure variation along the duct is neglected. To take the pressure variations in consideration, the flow field is described by parabolic Navier-Stokes equations [1]. Because of the nature of parabolic Navier-Stokes equations, the solution at certain position along the passage is dependent only on the boundary conditions of the problem and hence the numerical solution can be carried out in step by step manner. The dimensionless form of the governing equations are solved by local similarity-method [3].

2. GOVERNING EQUATIONS

Consider the laminar flow between two parallel plates as shown in Fig. (1). The uniform velocity of approach, the velocity at the axis of similarity, the pressure at inlet section and the distance between the two plates are denoted by u_o, $u_{o,x}$, p_o and 2b, respectively. The x-component and y-component of velocity are denoted as u and v, respectively.

The flow field is described through three governing partial differential equations; continuity equation, momentum equation in x-direction and momentum equation in y-direction. They can be written in cartesian co-ordinate as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad , \tag{2}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial y^2} \quad , \tag{3}$$

where ρ and ν are the denisty and the kinematic viscosity of the fluid, which are assumed to be constant through out the flow field. In the momentum equations the second derivative with respect to x are neglected compared with that with respect to y. The value of velocity at the axis of similarity ($u_{o,x}$) at any value of x must satisfy the continuity equation in integral form, which states :

$$\int_0^b u \, dy = u_o \, b \quad . \tag{4}$$

Because of the flow is similar about the axis of the passage, it is enough to solve the governing equations of the flow from one wall to the center of the passage, moreover the flow along the axis of passage is assumed to be potential flow and hence the relation between the velocity and pressure there; is described by Bernoulli's equation as follows :

$$\frac{p_{o,x} - p_o}{\rho} = \frac{u_o^2 - u_{o,x}^2}{2} \quad , \qquad (5)$$

where $p_{o,x}$ is the pressure at the center of the passage at general position x .

Equations (1)-(3) are a set of partial differential equations with the unkowns u, v and p. This set of equations must satisfies the following boundary conditions :

$$u = v = 0 \quad ; \quad \frac{\partial p}{\partial y} = \rho\nu \frac{\partial^2 v}{\partial y^2} \quad \text{at } y=0 \; , \qquad (6\text{-a})$$

$$u = u_{o,x} \quad ; \quad p = p_{o,x} \quad \text{at } y = b \; . \qquad (6\text{-b})$$

The boundary condition $\frac{\partial p}{\partial y})_{y=o} = \rho\nu \frac{\partial^2 v}{\partial y^2}$ is obtained by examining the second momentum equation (3).

To express the governing equations in dimensionless form, new independent variables ξ, η are interduced as follows:

$$\xi = \frac{1}{b}\sqrt{\frac{x\,\nu}{u_o}} \quad , \quad \eta = y\sqrt{\frac{u_o}{\nu\, x}} \quad . \qquad (7)$$

Furthermore, a dimensionless stream function and a dimensionless pressure are defined according to the following relations :

$$\psi(x,y) = \sqrt{u_o\, \nu\, x}\;\; f(\xi,\eta) \quad , \qquad (8\text{-a})$$

$$P(\xi,\eta) = (\; p(x,y) - p_o \;) \;/\; \rho\, u_o^2 \quad , \qquad (8\text{-b})$$

Where $\psi(x,y)$ is the stream function, which is defined such that it satisfies the continuity equation (1) and $P(\xi,\eta)$ is the dimensionless pressure. Substitution of equations (7)-(8) into equations (2)-(3) leads to the following dimensionless form of momentum equations (where, the primes denoting differentiation with respect to η) :

$$2\, f''' + f\, f'' + \eta\, P' = 0 \quad , \qquad (9)$$

$$2\, f''' + (f + \frac{2}{\eta})\, f'' + (f' - \frac{1}{\eta} f)\, f' - Re_b^2 \frac{\xi^2}{\eta} P' = 0 \quad . \qquad (10)$$

According to the local similarity method [3], the derivatives with respect to ξ in equations (9)&(10) are neglected and ξ is considered as a parameter in the equations. Re_b is the Reynolds number based on the half of the passage height ($Re_b = \frac{u_o\, b}{\nu}$). The continuity equation;

eqn (4) in dimensionless form is as follows :

$$\int_0^b f' \, d\eta = \eta_b \quad , \qquad (11)$$

where η_b is the value of η at the center of passage. According to equations (5) and (7)-(8) the dimensionless pressure at the center of the passage at any value of x (ξ) is given by

$$P(\xi,\eta_b) = \frac{1}{2} [\, 1- f_b'^2] \quad , \qquad (12)$$

where f_b' is the derivative of dimensionless stream function at the center of the passage.

Eliminating f"' from equations (9),(10) yields to the equation;

$$P' = \frac{2 f'' + f'(\eta f'' - f)}{4 \, Re_b^2 \, \xi^2 + \eta^2} \, . \qquad (13)$$

Equations (9) and (13) represent a system of ordinary differential equations in f and P as unknowns. Their boundary conditions can be deduced by examining equations (6) and (13) with aid of equations (5)-(8) as follows:

$$f = f' = 0 \quad , \quad P' = \frac{f''}{2 \, Re_b^2 \, \xi^2} \quad \text{at } \eta = 0 \, , \qquad (14\text{-a})$$

$$f' = \frac{u_{o,x}}{u_o} \quad , \quad P = \frac{1}{2} [1 - f'^2] \quad \text{at } \eta = \eta_b . \qquad (14\text{-b})$$

3. NUMERICAL PROCEDURE

For certain value of passage height (2b); which means certain value of Re_b, equations (9),(13) and (14) are solved for different values of the parameter ξ and hence for different values of η_b; where $\eta_b = \frac{1}{\xi}$ as it is shown in (7). At every value of ξ the set of equations is solved by Runge-Kutta numerical method of ordinary differential equations accompanied with shooting method of boundary value problems. To ensure the rapid convergence of solution, the calculation procedure is carried out through two main steps. First, the set of equations is solved for assumed f_b' and second step is to justify this value to produce a velocity profile satisfies the continuity equation; eqn (11). Table (1) showes the values used in numerical present calculations.

Knowing velocity field, the local coefficient of friction can be determined according to the following definition :

$$C_f = \tau_w \, / \, \rho \, u_o^2 \quad , \qquad (15)$$

where τ_w is the shear stress at the wall, which is defined by :

$$\tau_w = \rho \nu \frac{\partial u}{\partial y})_{y=0} \qquad (16)$$

Introducing the dimensionless variables in equations 15-16, one obtains the following expression of local coefficient of friction :

$$C_f \sqrt{Re_x} = f''(\xi, 0) \quad , \qquad (17)$$

where Re_x denotes the local Reynolds number ($\frac{u_o x}{\nu}$).

4. RESULTS AND DISCUSSION

The calculations are carried out for three different passage heights corresponding to the value of Re_b equals to 100, 200 and 300. The obtained numerical results are represented in the following figures. Fig.(2) shows the velocity profile in the main flow direction at different values of ξ for passage height corresponding to Re_b = 200. Near the entrance of the passage (ξ= 0.0714) the velocity is almost uniform except near the walls of the passage. With increasing value of ξ the profile is developing till ξ=0.2, there the velocity takes a profile near that of fully developed flow. The velocity at the center of the passage against ξ and x/b is shown in Fig. (3)&(4) respectively. As it is expected, Fig.(4), the velocity increases rapidly as the passage height is smaller or, in another word, the entrance length is shorter for narrow passages. Coefficient of friction is represented in Fig.(5)-(6). The coefficient, in general, as shown in figure (6) has higher values in case of Re_b = 100. It, for all values of Re_b, goes to asymptotic values. A summary of numerical results are tabulated in table (1).

5. CONCLUSION

The technique used in this work introduces a self starting method to solve the parabolic Navier-Stokes equations, which can be considered as a better approximation of the full Navier-Stokes equations in comparison with boundary layer approximation. According to the method used, the velocity profile and other properties of the flow at any position along the passage can be independently predicted. No more than the properties of flow at the entrance of the passage are required to carry out the solution at any position along the passage.

6. NOMENCLATURE

2b	the passage height
c_f	coefficient of friction , $\tau_w / \rho u_o^2$
f	dimensionless stream function , $\psi / \sqrt{u_o \nu x}$

f'_b	derivative of dimensionless stream function at the center of the passage , $u_{o,x}$ $/u_o$
Re_b	Reynolds number based on b , $u_o b/\nu$
Re_x	local Reynolds number , $u_o x/\nu$
u	velocity component in x-direction
u_o	the velocity at the inlet of the passage
$u_{o,x}$	the velocity at the center of the passage at any x
v	the velocity component in y-direction
x	co-ordinate along the lower wall of the passage
y	co-ordinate normal to the lower wall of the passage
η	dimensionless independent variable , $y\sqrt{u_o/x\,\nu}$
η_b	the value of η at the center of the passage
ξ	dimensionless independent variable , $\frac{1}{b}\sqrt{x\,\nu/u_o}$
ν	fluid kinematic viscosity
ρ	fluid density
$\tau_{w,x}$	wall shear stress in x-direction
ψ	stream function

7. REFERENCES

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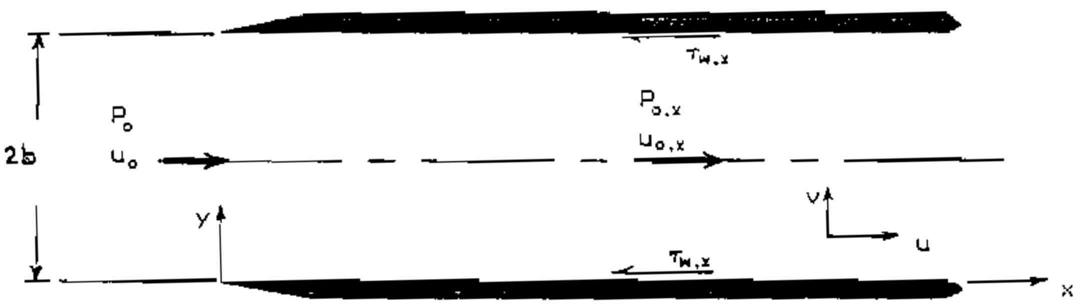

Fig. (1) Schematic description of the flow between two parallel plates

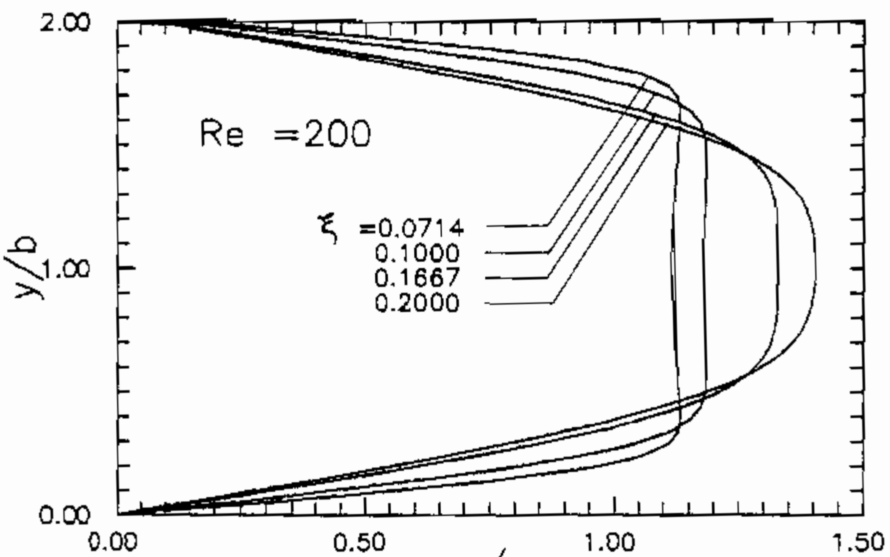

Fig. (2) The development of the velocity profile of laminar flow in the entrance region of the passage.

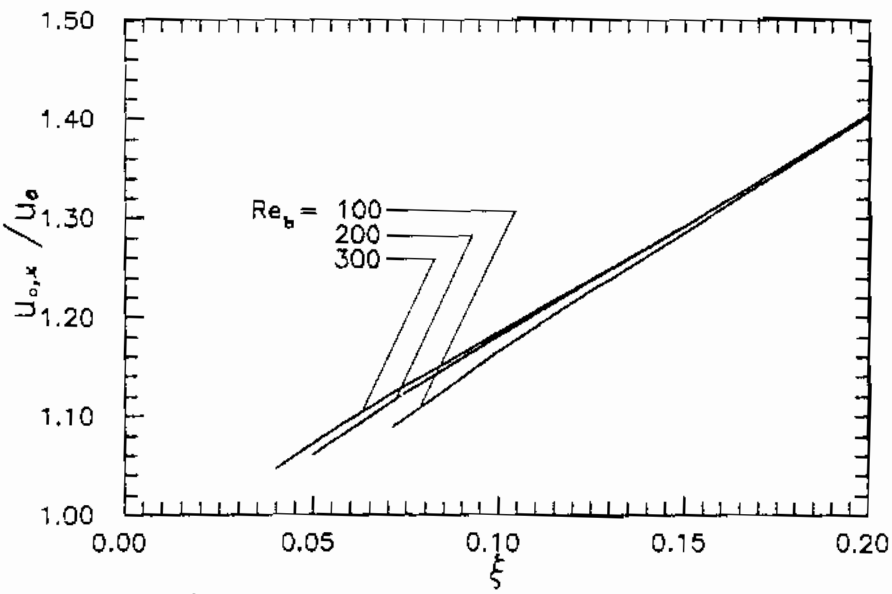

Fig. (3) The velocity at the center of the passage versus the dimensionless variable ξ.

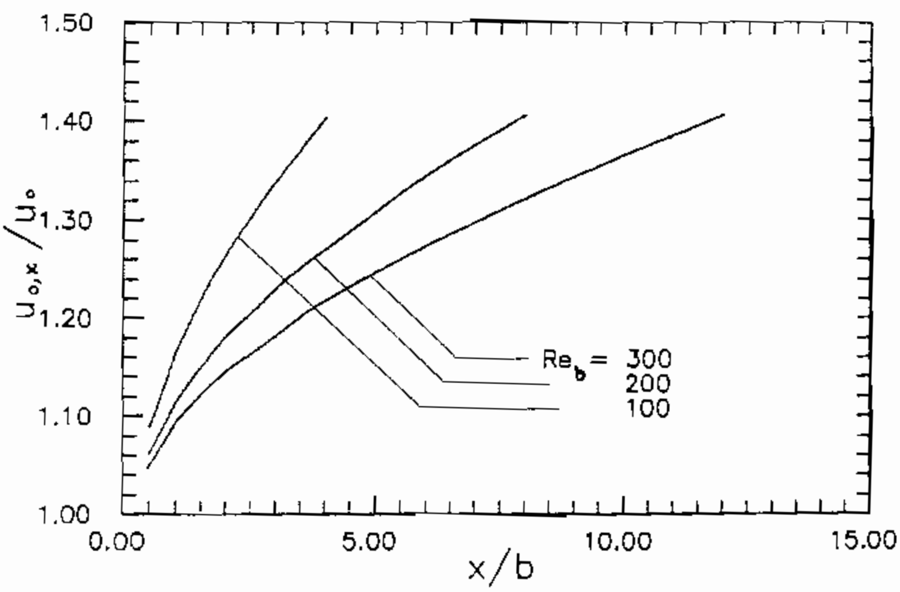

Fig. (4) The maximum velocity along the passage length.

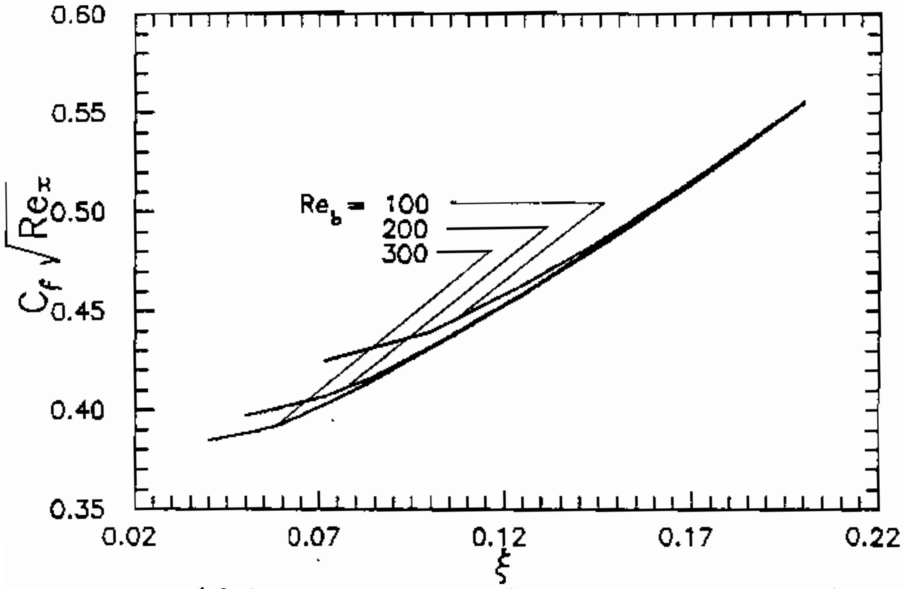

Fig. (5) The coefficient of friction versus the dimensionless variable ξ .

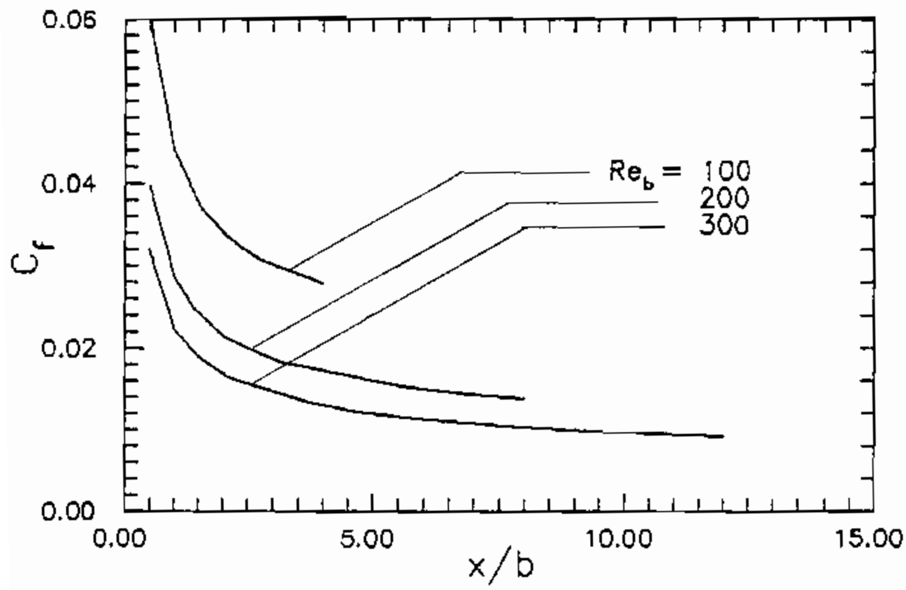

Fig. (6) The coefficient of friction along the passage length

Re_b	η_b	ξ	x/b	C_f
100	14.00	0.0714286	0.5102	0.05951
	10.00	0.100000	1.000	0.04400
	8.00	0.125000	1.5625	0.03707
	7.00	0.1428571	2.0408	0.03381
	6.00	0.166667	2.7778	0.03071
	5.00	0.200000	4.0000	0.02780
200	20.00	0.050000	0.500	0.039716
	14.00	0.0714286	1.0204	0.028474
	12.00	0.083333	1.3889	0.024987
	10.00	0.100000	2.0000	0.021622
	08.00	0.125000	3.1250	0.018382
	06.00	0.166667	5.5556	0.015309
	05.50	0.1818182	6.6112	0.014582
	05.00	0.200000	8.0000	0.013880
300	25.00	0.04000	0.4800	0.032079
	17.00	0.05882	1.0207	0.022247
	14.00	0.071429	1.5306	0.018793
	12.00	0.08333	2.0833	0.016556
	09.00	0.011111	3.7037	0.013291
	08.00	0.12500	4.6875	0.012235
	07.00	0.14286	6.12246	0.011201
	06.00	0.16667	8.3333	0.010201
	05.50	0.181819	9.91740	0.00972
	05.00	0.20000	12.000	0.00925

Table (1) The used values of the problem's parameters and the corresponding coefficient of friction .