يتألف الإختبار من 4 أسئلة في صفحتين. برجاء بدء إجابة كل فرع من إحدى نهايتي ورقة الإجابة.

Question (1) [30 points]

(a) Evaluate the following integrals [10 points]

(i) $\int_0^1 \sqrt{-\ln t}\,dt$ (ii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}dx$

(b) Given $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$, Evaluate the integral

$$\int_{-1}^{1} x\ P_n(x)P_{n-1}(x)dx$$ [1

points]

(c) Given the recurrence relation $\int x^n J_{n-1}(x)dx = x^n J_n(x) + c$, evaluate the integral

$$\int x^4 J_1(x)dx$$ [10 points]

Question (2) [25 points]

(a) Obtain the Fourier series of the function $f(x)$ defined by:

$f(x) = x^2$, on the interval $[0, 2\pi]$, $f(x) = f(x + 2\pi)$. [10 point

(b) Use the separation of variables technique to solve the heat equation

$$u_{xx} = \frac{1}{k}u_t, \quad 0 < x < L, \ t > 0$$

subjected to the conditions

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x)$$ [15 points]

3. a. [10 pts] Show that $u(x,y) = \cos x \sinh y$ is harmonic, find its harmonic conjugate $v(x,y)$, and express $u(x,y) + iv(x,y)$ as a function of z.

b. [10 pts] Solve $\sin z = 5$.

c. [10 pts] Expand $f(z) = \frac{1}{(z+2)(z-3)}$ in a Laurent series valid in

i. $2 < |z| < 3$.

ii. $0 < |z+2| < 5$.

4. a. [10 pts] Discuss the transformation $w = z^2$.

b. [10 pts] Without using the residue theorem, evaluate $\frac{1}{2\pi i}\int_{|z|=2a} \frac{ze^z}{(z-a)^3}\,dz$.

c. [10 pts] Use the residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4x+20}\,dx$.
