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# ALGEBRAIC PROPERTIES OF THE CURVATURE TENSOR ON A BANACH MANIFOLD

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## ABSTRACT

*For a finite dimentional Riemannian manifold, the curvature tensor has a number of interesting algebraic properties. In this paper, our aim is to generalize the concept of curvature tensor and study its properties for an infinite dimensional manifold modelled on a Banach space. Also, a generallization of the Ricci's identity ([2]) to the case of a Banach manifold is established.*

## INTRODUCITON

Let M be a Banach manifold of class $C^r$ $(r \geq 2, \infty)$ modelled on a Banach space E ([1]), and let $\bar{\Gamma}$ be the linear connection of class $C^{r-2}$ on M. For any x of M and any arbitrary chart $C=(U,\varphi,E)$ at x, let $p = \varphi(x) \in \varphi(U)$ and $\bar{h}_i \in T_X M$ (tangent space of M at x ([1])), $i=\overline{1, n+1}$.

Let $\bar{\alpha}$ and $\bar{A}$ be two tensors of rank $(0,n)$ and $(1,n)$ resp. on M, We define $\Gamma$, $\alpha$, A and $h_i$, $i=\overline{1, n+1}$ to be the models of $\bar{\Gamma}$, $\bar{\alpha}$, $\bar{A}$ and $\bar{h}_i$, resp., with respect to the chart C. In ([1]) the covariant differentiation of $\bar{\alpha}$ and $\bar{A}$ is given locally as follows.

$$\nabla \alpha_p (h_{n+1}; h_1,\dots, h_n) = D \alpha_p (h_{n+1}; h_1,\dots, h_n) - $$

$$- \sum_{i=1}^{n} \alpha_p (h_1,\dots, h_{i-1}, \Gamma_p (h_i, h_{n+1}), h_{i+1},\dots, h_n), \dots (1.1)$$

$$\nabla A_p (h_{n+1}; h_1,\dots, h_n) = D A_p (h_{n+1}; h_1,\dots, h_n) -$$

$$- \sum_{i=1}^{n} A_p (h_1,\dots, h_{i-1}, \Gamma_p (h_i, h_{n+1}), h_{i+1},\dots, h_n) +$$

$$\Gamma_p (A_p(h_1,\dots, h_n), h_{n+1}), \quad \dots\dots (1.2)$$

where D is the Frechet derivative (see([1])).

Also, in ([1]) the curvature tensor $\overline{R}_x$ and the torsion tensor $\overline{S}_x$ on M are simply given locally in the chart C as follows:

$$R_p (h_3; h_1, h_2) = D \Gamma_p (h_2; h_3, h_1) - D \Gamma_p (h_1; h_3, h_2) +$$

$$+ \Gamma_p(\Gamma_p(h_3,h_1),h_2)-\Gamma_p(\Gamma_p(h_3,h_2),h_1), \dots\dots (1.3)$$

$$S_p (h_1, h_2) = \frac{1}{2} (\Gamma_p (h_1, h_2) - \Gamma_p (h_2, h_1)) \dots\dots (1.4)$$

## 2. Ricci's identity:

Let $\overline{A}$ and $\overline{\alpha}$ be two tensors of type (1,n) and (0,n), respectively, on a Banach manifold M of class $C^r$ ($r \geq 2, \infty$) with a linear connection $\overline{\Gamma}$ of class $C^{r-2}$. Then $\forall\ x \in M$ and $\overline{h}_i \in T_X M$, $i = \overline{1, n+2}$.

$$\overline{\nabla} (\overline{\nabla}\, \overline{\alpha})_X (\overline{h}_{n+2}; \overline{h}_{n+1}; \overline{h}_1 \dots, \overline{h}_n) - \overline{\nabla} (\overline{\nabla}\, \overline{\alpha})_X (\overline{h}_{n+1}; \overline{h}_{n+2}; \overline{h}_1 \dots \overline{h}_n) =$$

$$= \sum_{i=1}^{n} \overline{\alpha}_X (\overline{h}_1,\dots, \overline{h}_{i-1}, \overline{R}_X (\overline{h}_i, \overline{h}_{n+2}, \overline{h}_{n+1}), \overline{h}_{i+1},\dots, \overline{h}_n) +$$

$$2\, \overline{\nabla}\, \overline{\alpha}_X (\overline{S}_X (\overline{h}_{n+2}, \overline{h}_{n+1}); \overline{h}_1,\dots, \overline{h}_n) \dots\dots (2.1)$$

$$\bar{\nabla}(\bar{\nabla}\bar{A})_X(\bar{h}_{n+2};\bar{h}_{n+1};\bar{h}_1,\ldots,\bar{h}_n) - \bar{\nabla}(\bar{\nabla}\bar{A})_X(\bar{h}_{n+1};\bar{h}_{n+2};\bar{h}_1,\ldots,\bar{h}_n) =$$

$$= \sum_{i=1}^{n} \bar{A}_X(\bar{h}_1,\ldots,\bar{h}_{i-1},\bar{R}_X(\bar{h}_i,\bar{h}_{n+2},\bar{h}_{n+1}),\bar{h}_{i+1},\ldots,\bar{h}_n) +$$

$$+\bar{R}_X(\bar{A}_X(\bar{h}_1,\ldots,\bar{h}_n);\bar{h}_{n+1},\bar{h}_{n+2}) +$$

$$+ 2\bar{\nabla}\bar{A}_X(\bar{S}_X(\bar{h}_{n+2},\bar{h}_{n+1});\bar{h}_1,\ldots,\bar{h}_n), \qquad (2.2)$$

where $\bar{R}$ and $\bar{S}$ are the curvature and torsion tonsors on M, respectively.

<u>Proof:</u>

To prove (2.1), (2.2), it sufficies to prove them locally with respect to an arbitrary chart on M. Let C = (U, $\varphi$, E) be a chart at the point $x \in M$ and R, S, $\Gamma$, $\alpha$, A, $h_i$; i= $\overline{1, n+2}$ are the models of $\bar{R}$, $\bar{S}$, $\bar{\Gamma}$, $\bar{\alpha}$, $\bar{A}$, $\bar{h}_i$, respectively, with respect to the chart C, $p = \varphi(x)$. Covariant differentiation of $\nabla\alpha$, which is defined in (1.1) yields

$$\nabla(\nabla\alpha)_p(h_{n+2};h_{n+1};h_1,\ldots,h_n) =$$

$$= D(\nabla\alpha)_p(h_{n+2};h_{n+1};h_1,\ldots,h_n) -$$

$$- \nabla\alpha_p(\Gamma_p(h_{n+1},h_{n+2});h_1\ldots,h_n) -$$

$$- \sum_{i=1}^{n} \nabla\alpha_p(h_{n+1};h_1,\ldots,h_{i-1},\Gamma_p(h_i,h_{n+2}),h_{i+1},\ldots,h_n) =$$

$$= D^2\alpha_p(h_{n+2};h_{n+1};h_1\ldots,h_n) -$$

$$- \sum_{i=1}^{n} D\alpha_p(h_{n+2};h_1,\ldots,h_{i-1},\Gamma_p(h_i,h_{n+1}),h_{i+1},\ldots,h_n) -$$

$$- \sum_{i=1}^{n} \alpha_p (h_1, \ldots, h_{i-1}, D\,\Gamma_p (h_{n+2}; h_i, h_{n+1}), h_{i+1}, \ldots, h_n) -$$

$$- \nabla \alpha_p (\Gamma_p (h_{n+1}, h_{n+2}); h_1 \ldots, h_n) -$$

$$- \sum_{i=1}^{n} D\,\alpha_p (h_{n+1}; h_1, \ldots, h_{i-1}, \Gamma_p (h_i, h_{n+2}), h_{i+1}, \ldots, h_n) +$$

$$+ \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_p (h_1, \ldots, h_{j-1}, \Gamma_p (h_j, h_{n+1}), \ldots, h_{i-1}, \Gamma_p (h_i, h_{n+2}), \ldots, h_n) +$$

$$+ \sum_{i=1}^{n} \alpha_p (h_1, \ldots, \Gamma_p (\Gamma_p (h_i, h_{n+2}), h_{n+1}), h_{i+1}, \ldots, h_n) \ldots (2.3)$$

By alternating both sides of formula (2.3) with respect to $h_{n+1}$ and $h_{n+2}$ and denoting this operation by underling them, we obtain

$$2\, \nabla (\nabla \alpha)_p (\underline{h}_{n+2}, \underline{h}_{n+1}; h_1, \ldots, h_n) =$$

$$= \sum_{i=1}^{n} \alpha_p (h_1, \ldots, h_{i-1}, D\,\Gamma_p (h_{n+1}; h_i, h_{n+2}) -$$

$$- D\,\Gamma_p (h_{n+2}; h_i, h_{n+1}) +$$

$$+ \Gamma_p (\Gamma_p (h_i, h_{n+2}), h_{n+1}) - \Gamma_p (\Gamma_p (h_i, h_{n+1}), h_{n+2}), h_{i+1}, \ldots h_n) +$$

$$+ \nabla \alpha_p (\Gamma_p (h_{n+2}, h_{n+1}) - \Gamma_p (h_{n+1}, h_{n+2}); h_1 \ldots, h_n) \ldots (2.4)$$

By using (1.3) and (1.4): the identity (2.4) takes the form:

$$2\, \nabla (\nabla \alpha)_p (\underline{h}_{n+2}; \underline{h}_{n+1}; h_1 \ldots, h_n) =$$

$$= \sum_{i=1}^{n} \alpha_p (h_1, \ldots, h_{i-1}, R_p (h_i ; h_{n+2}, h_{n+1}), h_{i+1}, \ldots, h_n) +$$

$$+ 2 \nabla \alpha_p (S_p (h_{n+2}, h_{n+1}); h_1 \ldots, h_n).$$

i.e. (2.1) holds.

Similarly, the covariant differentiation of $\nabla A$ with respect to $h_{n+2}$, which is defined in (1.2), yields

$$\nabla(\nabla A)_p (h_{n+2}; h_{n+1}; h_1, \ldots, h_n) = D (\nabla A)_p (h_{n+2}; h_{n+1}; h_1, \ldots, h_n) -$$

$$- \nabla A_p (\Gamma_p (h_{n+1}, h_{n+2}); h_1 \ldots, h_n) -$$

$$- \sum_{i=1}^{n} \nabla A_p (h_{n+1}; h_1, \ldots, h_{i-1}, \Gamma_p (h_i, h_{n+2}), h_{i+1}, \ldots, h_n) +$$

$$+ \Gamma_p (\nabla A_p (h_{n+1}; h_1, \ldots, h_n), h_{n+2}) =$$

$$= D^2 A_p (h_{n+2}, h_{n+1}; h_1 \ldots, h_n) -$$

$$- \sum_{i=1}^{n} D A_p (h_{n+2}; h_1, \ldots, h_{i-1}, \Gamma_p (h_i, h_{n+1}), h_{i+1}, \ldots, h_n) -$$

$$- \sum_{i=1}^{n} A_p (h_1, \ldots, h_{i-1}, D \Gamma_p (h_{n+2}; h_i, h_{n+1}), h_{i+1}, \ldots, h_n) +$$

$$+ D \Gamma_p (h_{n+2}; A_p (h_1, \ldots, h_n), h_{n+1}) +$$

$$+ \Gamma_p (D A_p (h_{n+2}; h_1, \ldots, h_n), h_{n+1}) - \nabla A_p (\Gamma_p (h_{n+1}, h_{n+2}); h_1 \ldots, h_n) -$$

$$- \sum_{i=1}^{n} D A_p (h_{n+1}; h_1, \ldots, h_{i-1}, \Gamma_p (h_i, h_{n+2}), h_{i+1}, \ldots, h_n) +$$

$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} A_p(h_1,\dots,\Gamma_p(h_j,h_{n+1}),\dots,\Gamma_p(h_i,h_{n+2}),\dots,h_{i+1},\dots,h_n)+$$

$$+\sum_{i=1}^{n} A_p(h_1,\dots,h_{i-1},\Gamma_p(\Gamma_p(h_i,h_{n+2}),h_{n+1}),h_{i+1},\dots,h_n)\ -$$

$$-\sum_{i=1}^{n} \Gamma_p(A_p\ (h_1,\dots,\ h_{i-1},\ \Gamma_p(h_i,\ h_{n+2}),\dots,h_n),\ h_{n+1})\ +$$

$$+\Gamma_p(DA_p(h_{n+1};h_1,\dots,h_n)-\sum_{i=1}^{n} A_p(h_1,\dots,h_{i-1},\Gamma_p(h_i,h_{n+1}),\dots,h_n)+$$

$$+\Gamma_p(A_p\ (h_1,\dots,h_n),h_{n+1}),h_{n+2}). \qquad \dots\dots(2.5)$$

Similary, alternating both sides of (2.5) with respect to $h_{n+2}$ and $h_{n+1}$ and using (1.3), (1.4) we have

$$2\ \nabla(\nabla\ A)_p\ (\underline{h}_{n+2};\ \underline{h}_{n+1};\ h_1,\dots,\ h_n) =$$

$$=\sum_{i=1}^{n} A_p\ (h_1,\dots,\ h_{i-1},\ R_p\ (h_i;h_{n+2},\ h_{n+1}),\ h_{i+1},\dots,\ h_n)\ +$$

$$+2\ \nabla\ A_p\ (S_p(h_{n+2},h_{n+1}):\ h_1,\dots,\ h_n)\ +$$

$$+\ 2\ D\ \Gamma_p\ (\underline{h}_{n+2};\ A_p\ (h_1,\dots,h_n),\ \underline{h}_{n+1})\ +$$

$$+\ 2\ \Gamma_p\ (\ \Gamma_p\ (A_p\ (h_1,\dots,h_n),\ \underline{h}_{n+1}),\ \underline{h}_{n+2}).$$

i.e. $2\ \nabla\ (\nabla A)_p\ (\underline{h}_{n+2}:\ \underline{h}_{n+1},h_1,\dots,h_n) =$

$$= \sum_{i=1}^{n} A_p (h_1, \dots, h_{i-1}, \underset{p}{R} (h_i; h_{n+2}, h_{n+1}), h_{i+1}, \dots, h_n) +$$
$$+ 2 \nabla A_p (\underset{p}{S} (h_{n+2}, h_{n+1}); h_1 \dots, h_n) +$$
$$+ \underset{p}{R} (A_p (h_1, \dots h_n); \underline{h}_{n+1}, \underline{h}_{n+2}).$$

Hence (2.2) holds and this completes the proof.■

**Remark.**

If the connection $\bar{\Gamma}$ is symmetric, then the torsion tensor is zero and therefore the terms which contain $\bar{S}$ in (2.1) and (2.2) vanish.

Now, let M be a Riemannian manifeld, then there is defined on M a tensor field $\bar{g}$ of type (0,2) which is symmetric and postive definite. We shall assume always that g is of class $C^{\infty}$.

**Theorem 2.1**

For any four tangent vectors $\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4 \in T_X M$ on a Riemannian manifold M, we have

$$\bar{g}_X(\bar{h}_4, \bar{R}_X(\bar{h}_3; \bar{h}_1, \bar{h}_2)) + \bar{g}_X(\bar{h}_3, \bar{R}_X(\bar{h}_4; \bar{h}_1, \bar{h}_2)) = 0 \quad (2.6)$$

$$\bar{g}_X(\bar{h}_4, \bar{R}_X(\bar{h}_3; \bar{h}_1, \bar{h}_2)) = \bar{g}_X(\bar{h}_1, \bar{R}_X(\bar{h}_2; \bar{h}_4, \bar{h}_3)), \quad (2.7)$$

Where $\bar{R}_X$ and $\bar{g}_X$ are the curvature tensor and the metric tensor ([2]) on M, respectively.

**Proof.**

Since M is a Riemannian manifold, then the fundamental meteric tensor $\bar{g}$ is covariant constant ([2]), that is

$$\bar{\nabla} \bar{g} = 0. \quad \dots\dots\dots\dots\dots\dots (2.8)$$

As the metric g is covariant tensor of type (0,2), then from (2.1) we have

$$\bar\nabla(\bar\nabla\bar g)_X(\bar h_1;\bar h_2,\bar h_4\ \bar h_3)-\bar\nabla(\bar\nabla\bar g)_X(\bar h_2;\bar h_1;\bar h_4,\bar h_3)=$$
$$=\bar g_X(\bar h_4,\bar R_X(\bar h_3;\bar h_1,\bar h_2))+\bar g_X(\bar h_3,\bar R_X(\bar h_4;\bar h_1,\bar h_2))+$$
$$+2\bar\nabla\bar g_X(\bar S_X(\bar h_1;\bar h_2);\bar h_4\ \bar h_3) \qquad \ldots\ldots\ldots\ldots\ldots(2.9)$$

Substituting from (2.8) in (2.9) we obtain

$$\bar g_X(\bar h_4,\bar R_X(\bar h_3;\bar h_1,\bar h_2))+\bar g_X(\bar h_3,\bar R_X(\bar h_4;\bar h_1,\bar h_2))=0,$$

and this proves (2.6).

By using

$$\bar R_X(\bar h_3;\bar h_1,\bar h_2)=-\bar R_X(\bar h_3;\bar h_2,\bar h_1),$$

$$\bar R_X(\bar h_3;\bar h_1,\bar h_2)+\bar R_X(\bar h_1;\bar h_2,\bar h_3)+\bar R_X(\bar h_2;\bar h_3,\bar h_1)=0$$

and (2.6) we can verify that:

$$\bar g_X(\bar R_X(\bar h_3;\bar h_2,\bar h_4),\bar h_1)+\bar g_X(\bar R_X(\bar h_2;\bar h_1,\bar h_4),\bar h_3)+$$

$$+\bar g_X(\bar R_X(\bar h_3;\bar h_1,\bar h_2),\bar h_4)=0. \qquad \ldots\ldots\ldots\ldots(2.10)$$

Consequently, we have that

$$\bar g_X(\bar R_X(\bar h_1;\bar h_2,\bar h_3),\bar h_4)+\bar g_X(\bar R_X(\bar h_4;\bar h_1,\bar h_3),\bar h_2)+$$
$$+\bar g_X(\bar R_X(\bar h_3;\bar h_1,\bar h_2),\bar h_4)=0. \qquad \ldots\ldots\ldots\ldots\ldots(2.11)$$

$$\bar g_X(\bar R_X(\bar h_3;\bar h_2,\bar h_4),\bar h_1)+\bar g_X(\bar R_X(\bar h_1;\bar h_2,\bar h_3),\bar h_4)+$$

$$+\bar g_X(\bar R_X(\bar h_2;\bar h_4,\bar h_3),\bar h_1)=0. \qquad \ldots\ldots\ldots\ldots(2.12)$$

and

$$\bar g_X(\bar R_X(\bar h_2;\bar h_1,\bar h_4),\bar h_3)+\bar g_X(\bar R_X(\bar h_4;\bar h_1,\bar h_3),\bar h_2)+$$

$$+\bar g_X(\bar R_X(\bar h_2;\bar h_4,\bar h_3),\bar h_1)=0. \qquad (2.13)$$

From (2.10) , (2.11) , (2.12) and (2.13) we have

$$2\bar g_X(\bar h_3,\bar R_X(\bar h_1;\bar h_2,\bar h_4))=2\bar g_X(\bar h_2,\bar R_X(\bar h_4;\bar h_3,\bar h_1)).$$

i.e.(2.7) is true and thus the proof of the theroem is complete.

Proposition 2.2.

Let M be Banach manifold of class $C^r$ $(r \geq 3, \infty)$ such that there exist two linear connections $\overset{1}{\Gamma}, \overset{2}{\Gamma}$ on M. Then $\forall\, x \in M$ and $\forall\, \bar{h}_1, \bar{h}_2, \bar{h}_3 \in T_X M$

$$\overset{2}{R}_X(\bar{h}_3; \bar{h}_1, \bar{h}_2) = \overset{1}{R}_X(\bar{h}_3; \bar{h}_1, \bar{h}_2) + \overset{1}{\nabla}\bar{T}_X(\bar{h}_2; \bar{h}_3, \bar{h}_1) -$$

$$-\overset{1}{\nabla}\bar{T}_X(\bar{h}_1; \bar{h}_3, \bar{h}_2) + \bar{T}_X(\bar{T}_X(\bar{h}_3, \bar{h}_1), \bar{h}_2) - \bar{T}_X(\bar{T}_X(\bar{h}_3, \bar{h}_2), \bar{h}_1) +$$

$$+ 2\,\bar{T}_X(\bar{h}_3, \overset{1}{S}_X(\bar{h}_1, \bar{h}_2)), \quad \text{..........................} (2.14)$$

where $\overset{1}{R}_X, \overset{2}{R}_X$ are the curvature tensors of the linear connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$, respectively, $\bar{T} = \overset{2}{\Gamma} - \overset{1}{\Gamma}$ is the tensor of deformations, $\overset{1}{\nabla}$ is the covariant differentiation with respect to $\overset{1}{\Gamma}$, and $\overset{1}{S}$ is the torsion tensor of $\overset{1}{\Gamma}$

**Proof:**

It sufficies to prove (2.14) locally with respect to an arbitrary chart C= (U, $\varphi$, E) at $x \in M$. Let $\overset{1}{R}, \overset{2}{R}, T, \overset{1}{S}, \overset{1}{\Gamma}, \overset{2}{\Gamma}$ and $h_i$, $i = \overline{1, 3}$ be the models of $\overset{1}{R}, \overset{2}{R}, \bar{T}, \overset{1}{S}, \overset{1}{\Gamma}, \overset{2}{\Gamma}$ and $\bar{h}_i$, respectively, with respect to the chart C, and let

$p = \varphi(x) \in \varphi(U)$

We denote the alternation of any two vectors by underling them.

From the definition of the curvature tensor, we have

$$\overset{2}{R}_p(h_3;\, h_1,\, h_2) = 2\, D\, \overset{2}{\Gamma}_p(\underline{h_2};\, h_3,\, \underline{h_1}) + 2\overset{2}{\Gamma}_p(\overset{2}{\Gamma}_p(h_3,\, \underline{h_1}),\, \underline{h_2}).$$

Then substituting $\overset{2}{\Gamma}_p = T_p + \overset{1}{\Gamma}_p$ in the above formula, we obtain

$$\overset{2}{R}_p(h_3;\, h_1,\, h_2) = \overset{1}{R}_p(h_3;\, h_1,\, h_2) + 2\, DT_p(\underline{h_2};\, h_3,\, \underline{h_1}) +$$

$$+ 2T_p(T_p(h_3,\, \underline{h_1}),\, \underline{h_2}) + 2T_p(\overset{1}{\Gamma}_p(h_3,\, \underline{h_1}),\, \underline{h_2}) +$$

$$+ 2\, \overset{1}{\Gamma}_p(T_p(h_3,\, \underline{h_1}),\, \underline{h_2}). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.15)$$

Adding and subtracting the term $2\, T(h_3,\, \overset{1}{\Gamma}_p(\underline{h_1},\, \underline{h_2}))$ to the right hand side of (2.15), and taking into account that $\bar{T}$ is a tensor of type (1, 2) on M and $\overset{1}{S}_p(h_1,\, h_2) = \overset{1}{\Gamma}_p(\underline{h_1},\, \underline{h_2})$ we obtain

$$\overset{2}{R}_p(h_3;\, h_1,\, h_2) = \overset{1}{R}_p(h_3;\, h_1,\, h_2) + 2\, \overset{1}{\nabla}\, T_p(\underline{h_2};\, h_3,\, \underline{h_1}) +$$

$$+ 2T_p(T_p(h_3,\, \underline{h_1}),\, \underline{h_2}) + 2T_p(h_3,\, \overset{1}{S}(h_1,\, h_2)).$$

and this proves (2.14).

<u>Theorem 2.3:</u> (Bianchi's identity)

If $\bar{R}$ be the curvature tensor of the linear connection $\bar{\Gamma}$ on a Banach manifold M of class $C^r$ $(r \geq 4,\, \infty)$. Then for any $x \in M$ and $\forall\ \bar{h}_i \in T_x M$, $i=\overline{1,4}$, we have:

$$\overline{\nabla R}_x(\bar{h}_1;\bar{h}_4;\bar{h}_2,\bar{h}_3) + \overline{\nabla R}_x(\bar{h}_2;\bar{h}_4;\bar{h}_3,\bar{h}_1)+\overline{\nabla R}_x(\bar{h}_3;\bar{h}_4;\bar{h}_1,\bar{h}_2)=0 \quad ..................(2.16)$$

**Proof:**

It sufficies to prove the identity locally with respect to an arbitrary chart $C = (U, \varphi, E)$ at arbitrary point $x \in M$.
For this let $p = \varphi(x) \in \varphi(U)$ and $R$, $\Gamma$, $h_i; i=\overline{1,4}$ be the models of $\bar{R}$, $\bar{\Gamma}$, $\bar{h}_i$ with respect to the chart $C$ .

Covariant differentiation of $R$ at the point $p$ which is defined in (1.3) gives us

$$\nabla R_p(h_3;h_4;h_1,h_2) = D\, R_p(h_3;h_4;h_1,h_2) - R_p(\Gamma_p(h_4,h_3);h_1, h_2) -$$
$$- R_p(h_4; h_1, \Gamma_p(h_2, h_3)) - R_p(h_4; \Gamma_p(h_1, h_3), h_2) +$$
$$+ \Gamma_p(R_p(h_4;h_1, h_2), h_3) =$$

$$= 2D^2\, \Gamma_p(h_3;\underline{h_2}; h_4, \underline{h_1}) + 2\, D\, \Gamma_p(h_3;\Gamma_p(h_4, \underline{h_1}), \underline{h_2}) +$$
$$+ 2\, \Gamma_p(D\, \Gamma_p(h_3;h_4, \underline{h_1}), \underline{h_2}) - R_p(\Gamma_p(h_4, h_3);h_1, h_2) -$$
$$- 2\, R_p(h_4; \underline{h_1}, \Gamma_p(\underline{h_2}, h_3)) + \Gamma_p(R_p(h_4;h_1, h_2),h_3) \quad ..........(2.17)$$

By a similar way we obtain

$$\nabla R_p(h_2;h_4;h_3,h_1) =$$
$$= 2D^2\, \Gamma_p(h_2;\underline{h_1}; h_4, \underline{h_3}) + 2\, D\, \Gamma_p(h_2;\Gamma_p(h_4, \underline{h_3}), \underline{h_1}) +$$
$$+ 2\, \Gamma_p(D\, \Gamma_p(h_2;h_4, \underline{h_3}), \underline{h_1}) - R_p(\Gamma_p(h_4, h_2);h_3, h_1) -$$
$$- 2\, R_p(h_4; \underline{h_3}, \Gamma_p(\underline{h_1}, h_2)) + \Gamma_p(R_p(h_4;h_3, h_1), h_2) \quad .........(2.18)$$

and

$$\nabla R_p(h_1; h_4; h_2, h_3) =$$
$$= 2D^2\, \Gamma_p(h_1;\underline{h_3}; h_4, \underline{h_2}) + 2\, D\, \Gamma_p(h_1;\Gamma_p(h_4, \underline{h_2}), \underline{h_3}) +$$

$$+ 2\,\Gamma_p\,(D\,\Gamma_p\,(h_1;h_4,\ \underline{h_2}),\ \underline{h_3}) - R_p\,(\Gamma_p\,(h_4,\ h_1);h_2,\ h_3) -$$
$$- 2\,R_p\,(h_4;\ \underline{h_2},\ \Gamma_p\,(\underline{h_3},\ h_1)) + \Gamma_p\,(R_p(h_4;h_2,\ h_3),\ h_1)$$

........................(2.19)

Substituting from (2.17), (2.18) and (2.19) in (2.16) we obtain

$$\nabla R_p\,(h_3;\ h_4;\ h_1,\ h_2) + \nabla R_p\,(h_2;\ h_4;\ h_3,\ h_1) +$$
$$+\nabla R_p\,(h_1;\ h_4;\ h_2,\ h_3) =$$
$$= 2\,\Gamma_p\,(D\,\Gamma_p\,(h_3;h_4,\ \underline{h_1}),\ \underline{h_2}) + 2\,\Gamma_p\,(D\,\Gamma_p(h_2;h_4;\ \underline{h_3}),\ \underline{h_1}) +$$
$$+ 2\Gamma_p(D\,\Gamma_p\,(h_1;h_4,\ \underline{h_2}),\ \underline{h_3}) + 2\,\Gamma_p(\Gamma_p(\Gamma_p(h_4,\ h_3),\ \underline{h_2}),\ \underline{h_1}) +$$
$$+ 2\Gamma_p(\Gamma_p\,(\Gamma_p(h_4,\ h_2),\ \underline{h_1}),\ \underline{h_3}) + 2\Gamma_p(\Gamma_p(\Gamma_p(h_4,\ h_1),\ \underline{h_3}),\underline{h_2}) +$$
$$+ \Gamma_p\,(R_p(h_4;h_1,\ h_2),\ h_3) + \Gamma_p\,(R_p(h_4;h_3,\ h_1),\ h_2) +$$
$$+ \Gamma_p\,(R_p(h_4;h_2,\ h_3),\ h_1) =$$

$$= \Gamma_p\,(2\,D\,\Gamma_p\,(\underline{h_{21}};h_4,\ \underline{h_{32}}) + 2\,\Gamma_p\,(\Gamma_p(h_4,\ \underline{h_3}),\ \underline{h_2}),\ h_1) +$$
$$+ \Gamma_p\,(2\,D\,\Gamma_p\,(\underline{h_3};h_4,\ \underline{h_1}) + 2\,\Gamma_p\,(\Gamma_p(h_4,\ \underline{h_1}),\ \underline{h_2}),\ h_2) +$$
$$+ \Gamma_p\,(2\,D\,\Gamma_p\,(\underline{h_1};h_4,\ \underline{h_2}) + 2\,\Gamma_p\,(\Gamma_p(h_4,\ \underline{h_2}),\ \underline{h_1}),\ h_3) +$$
$$+ \Gamma_p\,(R_p(h_4;h_1,\ h_2),\ h_3) + \Gamma_p\,(R_p(h_4;h_3,\ h_1),\ h_2) +$$
$$+ \Gamma_p\,(R_p(h_4;h_2,\ h_3),\ h_1) \ldots\ldots$$

......(2.20)

Using the linearity of the connection $\Gamma_p$ and the definition of $R_p$ and taking into account that $R_p$ is skew-symmetric with respect to the second and the third arguments, then (2.20) becomes

$$\nabla R_p\,(h_3;\ h_4;\ h_1,\ h_2) + \nabla R_p\,(h_2;\ h_4;\ h_3,\ h_1) +$$
$$+\nabla R_p\,(h_1;\ h_4;\ h_2,\ h_3) =$$
$$= \Gamma_p\,(R_p(h_4;h_1,\ h_3) + R_p(h_4;h_3,\ h_1),\ h_2) +$$
$$+ \Gamma_p\,(R_p(h_4;h_3,\ h_2) + R_p(h_4;h_2,\ h_3),\ h_1) +$$
$$+ \Gamma_p\,(R_p(h_4;h_2,\ h_1) + R_p(h_4;h_1,\ h_2),\ h_3) = 0.$$

and this completes the proof. ■

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## البحث الثانى: (بحث منفرد)

### اسم البحث:

**Algebraic Properties of the Curvature tensor on a Banach Manifold**

**الخواص الجبرية لممتد الانحناء على فراغات بناخ المتعددة الطيات الانهائية البعد**

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مجلة كلية العلوم -جامعة المنوفية -مصر

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