A NUMERICAL SOLUTION OF INTEGRAL EQUATIONS WITH A LOGARITHMIC KERNEL BY PRODUCT AND GAUSS-LEGENDRE METHODS

حل المعادلات التكاملية ذات النواة اللوغارتيمية عدديا باستخدام طرق المضروبات وجاوس - لاجندر

By

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الخلاصة: فى هذا البحث استخدمت طريقة المضروبات لحل معادلات فريدهولم من النوع الاول ذات النواة اللوغارتيمية عدديا , واقترحت كيفية زيادة دقة هذه الطريقة مع المقارنة بطريقة جاوس - لاجندر بعد تحسين دقتها باضافة بارامتر صغير ε_k

Abstract : An integral equation whose kernel presents logarithmic singularity is numerically solved by product method. Gauss-Legendre method is developed to be suitable in handling such type of integral equation. Hence, it is used to obtain a numerical solution for integral equation whose kernel presents logarithmic singularity A comparison and both the weakness and strengths of the solutions obtained by these methods are evaluated and discussed.

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1. Introduction : The general form of an integral equation whose kernel presents a logarithmic singularity is

$$\int_{-1}^{1} w(t)K(t,x)g(t)dt = f(x),\ x\ (1 \qquad \text{(1.a)}$$

with

$$K(t,x) = A(t,x)\ \ln|t-x| + B(t,x) \qquad \text{(1.b)}$$

where $A(t,x)$, $B(t,x)$ and $f(x)$ are regular in $[-1,1]$. This integral equation appears frequently in applied mathematics and physics. Some problems where this class of equations is encountered are mentioned by Morland[1], and Christiansen[2]. Wave scattering and heat transfer in strips problems and elasticity problems are also some sources of such applications which lead to these cases of singular integral equations of the first kind with logarithmic singularities in the kernels.

For some special cases of the weight function $w(t)$ and the coefficient function $A(t,x)$, analytical methods can be used to obtain the solution of this type of singular integral equation. Numerical methods have been proposed by many authors. A wide overview of such methods can be found in Delves and Walsh[3], Baker[4], Atkinson[5] and in Moss and Christensen[6] who themselves proposed such a Galerkin type method. A modification of this latter method has been proposed by Frenkel[7,8]. On the other hand Theocaris et al.[9] have extended to the case of the logarithmic kernel the direct methods used for Cauchy type singular integral equations and based on the collocation at appropriately selected points x_k. The method is very simple and its performance is reasonable, if one takes into account that the integrand of the reduced equation was bounded but its derivative was not. As mentioned by Chrysakis and Tsamasphyros[10], Ioakimidis[11] has considered the case $A(t,x)$ = constant and proposed the reduction of (1) to a Cauchy type equation by differentiation with respect to x, i.e.

$$\int_{-1}^{1} w(t)\, K^{*}(t,x)\, g(t)\, dt = f^{*}(x), \qquad \text{(2.a)}$$

where,

$$K^{*}(t,x) = \frac{-1}{t-x} + B^{*}(t,x) \qquad \text{(2.b)}$$

and solution of the latter by the classical methods[12,13] performing collocation of appropriately chosen points x_k. But, as it is known, this differentiation causes the loss of two constants in $K(t,x)$ and in $f(x)$, hence additional analysis is developed to ensure that the solution $g(t)$ of the last equation (2) satisfies the original equation (1) too. Chrysakis and Tsamasphyros[10] proposed another direct method for equation (1) with $A(t,x) = 1$. This method has the advantage of Ioakimidis[11] method (i.e., it has polynomial accuracy of degree 2n-1 by a collocation at n points, and hence reducing the problem to an (n x n) linear system). But in addition it does not need to perform the collocation at appropriately determined points x_k but at arbitrarily chosen ones $\bar{x}_k$.

In the present work, product method is applied to solve singular integral equations of the first kind with logarithmic singularity in their kernels along the integration interval. The technique used here is very simple and it has the advantage that it can be used on uniform and graded meshes or on Gauss's points. It can be also used as open type or closed type formula according to the choice of the nodes. Furthermore, a simple modification is introduced on the conventional Gauss-Legendre method to improve the accuracy of the obtained numerical solutions for the integral equations whose kernel presents logarithmic singularity

2. **Product Method** : Integral which contains logarithmic singularity can be evaluated by using product -2 as follows :

$$\int_a^b f(t)\ln|t-x|dt = P_1 f(t_1) + P_2 f(t_2), \qquad \text{(3)}$$

where $t_1, t_2 \in [a,b]$ and $f(t)$ is a regular function on the interval $[a,b]$. It is known that the weight functions $P_1(x)$ and $P_2(x)$ can be determined from the solution of the following system of equations :

$$\int_a^b t^k \ln|t-x|dt = P_1(x)t_1^k + P_2(x)t_2^k, \quad k = 0, 1 \qquad \text{(4)}$$

where the error in this method is of order h^3 ($O(h^3)$). As mentioned above the points t_1 and t_2 are arbitrarily chosen points which belong to the interval of integration $[a,b]$. Therefore, the following relations can be assigned to them :

$$\left.\begin{aligned} t_1 &= a + \theta(b-a) \\ t_2 &= b - \theta(b-a) \end{aligned}\right\} \qquad \text{(5)}$$

where $0 \le \theta < 0.5$.

Also, the effect of uniform and graded meshes on the obtained solution by this method may be taken into consideration, and hence we take :

$$x_k = \left(\frac{k}{N}\right)^{\beta}, \ k = 0, 1, 2, \ldots, N. \qquad \text{(6)}$$

where N is the number of subintervals, $x_{-k} = x_k$ and $0 < \beta < 2$.

3. **Gauss–Legendre Method** : Although the integral $\int_a^b \ln|t-x|dt$ and $|x| < 1$, has a singularity along the line x=t, it is convergent to a finite exact value, i.e.

$$\int_a^b \ln|t-x|dt = (b-x)\ln|b-x| - (a-x)\ln|a-x| - (b-a) \qquad \text{(7)}$$

But when using Gauss-2 method, the considered integral may have the following value:

$$\int_a^b \ln|t-x|dt = \frac{b-a}{2}\left[\ln|t_1-x| \ |t_2-x|\right] \qquad \text{(8)}$$

where, t_1 and t_2 are given in (5). The right hand side of (8) may lead to an infinite value if t_1 or t_2 is equal to the value of the variable x. This infinite value can be avoided. For instance, let $t_1 - x = \varepsilon \neq 0$ and hence the value of ε can be adjusted such that the value of the integral $\int_a^b \ln|t-x|\,dt$ obtained by Gauss-2 is equal to the exact value in (7). This value can be easily determined and is found to be :

$$\varepsilon_k = \frac{\sqrt{3}}{h_k}(c_1 h_k)^{2c_1}(\theta_G h_k)^{2\theta} e^{-2}, \quad k = 0, 1, \ldots, N$$

where

$$c_1 = \frac{3+\sqrt{3}}{6}, \qquad \theta_G = \frac{3-\sqrt{3}}{6} \quad \text{and}$$

$$h_k = x_{k+1} - x_k$$

4. Examples

4-1 Example (1) :

$K(x,t) = \ln|t-x|$

$f(x) = (1-x)\ln|1-x| + (1+x)\ln|1+x| - 2,$

$g(t) = 1$

Method N	Gauss-2		Product-2	
	$E_{r.m.s.}$	Time in sec.	$E_{r.m.s}$	Time in sec.
4	$1.98*10^{-3}$	1	$9.21*10^{-12}$	1
8	$1.39*10^{-3}$	5	$2.93*10^{-11}$	3
16	$9.83*10^{-4}$	27	$5.12*10^{-11}$	17
24	$8.07*10^{-4}$	75	$1.14*10^{-10}$	58

where $E_{r.m.s.}$ is the root mean square error and **Time in sec.** is the time of computations.

4-2 Example (2)

$K(x,t) = (x^2+t^2)\ln|t-x|,$

$$f(x) = \frac{1}{4}(2x^2 + 1 - 3x^4)\ln\left|\frac{1-x}{1+x}\right| - \frac{x}{6}(9x^2 + 1),$$

$g(t) = t$

Method N	Gauss-2		Product-2	
	$E_{r.m.s.}$	Time in sec	$E_{r.m.s}$	Time in sec.
4	$1.93*10^{-2}$	1	$4.99*10^{-2}$	1
8	$1.54*10^{-2}$	6	$1.87*10^{-2}$	3
16	$2.42*10^{-2}$	27	$6.96*10^{-3}$	19
24	$3.15*10^{-2}$	72	$3.89*10^{-3}$	57

But we can improve the accuracy of product-2 by one method of the following :

(i) The values of regular part of the kernel $K(t,x)$ is taken into consideration when the weights are evaluated, i.e. .

$$\int_a^b f(t)\,(t^2 - x^2)\,\ln|t-x|\,dt = P_1(x)f(t_1) + P_2(x)f(t_2), \qquad t_1, t_2 \in [a,b] \ldots\ldots\ldots (9)$$

The case study shows the effect of using this formula if it is compared with previous table.

N	$E_{r.m.s.}$	Time in sec.
4	$2.26*10^{-11}$	1
8	$9.94*10^{-11}$	6
16	$1.59*10^{-10}$	31
24	$1.76*10^{-9}$	86

(ii) By using graded meshes, i.e. :

$$x_k = \left(\frac{k}{N}\right)^{\beta} \qquad \beta \in (0.5, 2)$$

The effect of β on the solution is given in the following table for N=8

β	0.7	0.8	0.9	1.0	1.1
$E_{r.m.s}$	$3.48*10^{-2}$	$2.87*10^{-2}$	$2.34*10^{-2}$	$1.87*10^{-2}$	$1.43*10^{-2}$
β	1.2	1.3	1.4	1.5	1.6
$E_{r.m.s}$	$9.90*10^{-3}$	$5.24*10^{-3}$	$3.96*10^{-3}$	$1.19*10^{-2}$	$2.54*10^{-2}$

(iii) By changing the position of the points t_1, t_2 :

$t_1 = a + \theta\,(b-a)$

$t_2 = b - \theta\,(b-a)$

It is found that $\theta \in (0.2, 0.23)$

5- Conclusions : It can be concluded that the product method and Gauss - Legendre method can be used to solve integral equations whose kernels present logarithmic singularities. The product-2 method is a straight-forward technique which can be easily used to solve such type of integral equations. The accuracy of this method can be improved by:

a) Taking into consideration the regular part of the kernel.

b) Using graded mesh and

c) Changing the position of t_1 and t_2.

It is possible to use Gauss-2 method for solving integral equations whose kernels present logarithmic singularities avoiding the infinite values which may be appeared by using the adjusting values ε. Although, these adjusting values enable us to obtain a solution with accepted accuracy, they have a disadvantage. This disadvantage forms the only weakness in applying Gauss-2 method for solving this type of integral equation. The disadvantage is that ε depends only on the interval of integration and not on the integrand function

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