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Hankel Determinant for Class of Analytic functions involving Generalized Noor Integral Operator

G. E. Abo Elyazyd [1] *A. M. Shahin* [2] *and H. E. Darwish* [3]

[1,2,3] *Department of Mathematics Faculty of Science , Mansoura University Mansoura, 35516, Egypt.*

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Abstract: he objective of this paper is to obtain an upper bound to the second Hankel determinant $|u_2u_4 - u_3^2|$ for class of functions $\mathcal{F}(v) = v + \sum_{n=2}^{\infty} u_n v^n$ involving generalized Noor integral operator, which we denote by $\mathcal{S}_\lambda^\mu[\mathcal{I}_\lambda(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,p}]$.

keywords Fox-Wright, Hypergeometric, Convolution, Analytic, Integral.

1.Introduction

Let $\mathcal{A}$ denote the class of functions $\mathcal{F}(v)$ of the form

$$\mathcal{F}(v) = v + \sum_{n=2}^{\infty} u_n v^n \qquad (1)$$

which are analytic in the open unit disk

$$\mathcal{X} := \{v \in \mathbb{C} : |v| < 1\}.$$

By $\mathcal{S}, \mathcal{C}, \mathcal{S}^*, \mathcal{C}(\beta)$ and $\mathcal{S}^*(\beta)$ we denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, convex, starlike, convex of order β and starlike of order β in $\mathcal{X}$ respectively(see [5]). For the functions $\mathcal{F}(v)$ in (1) and

$$\mathcal{G}(v) = v + \sum_{n=2}^{\infty} b_n v^n, v \in \mathcal{X}, \qquad (2)$$

the convolution (Hadmard oproduct) is defined as

$$(\mathcal{F} * \mathcal{G})(v) = (\mathcal{G} * \mathcal{F})(v)$$

$$= v + \sum_{n=2}^{\infty} u_n b_n v^n, v \in \mathcal{X}.$$

Denoted by $d^\lambda : \mathcal{A} \to \mathcal{A}$ the operator

$$d^\lambda \mathcal{F}(v) := \frac{v}{(1-v)^{\lambda+1}} * \mathcal{F}(v), \qquad (3)$$

$\lambda > -1$.

This implies that

$$d^n \mathcal{F}(v) = \frac{v(v^{n-1}\mathcal{F}(v))^{(n)}}{n!}, \quad (4)$$

$$n \in N_0 = N \cup \{0\}.$$

The operator $d^n \mathcal{F}(v)$ is called Ruscheweyh derivative of n −th order of $\mathcal{F}(v)$. Note that the class $d^0 \mathcal{F}(v) = \mathcal{F}(v)$ and $d^1 \mathcal{F}(v) = v\mathcal{F}'(v)$. Recently, Noor [16, 18] defined and studied an integral operator $\mathcal{I}_n : \mathcal{A} \to \mathcal{A}$ analogous to $d^n \mathcal{F}(v)$ as follows.

Let $\mathcal{F}_n(v) = v/(1-v)^{n+1}$, $n \in N_0$, and let $\mathcal{F}_n^{(-1)}(v)$ be defined such that

$$\mathcal{F}_n(v) * \mathcal{F}_n^{(-1)}(v) = \frac{v}{1-v}. \qquad (5)$$

Then

$$\mathcal{I}_n \mathcal{F}(v) = \mathcal{F}_n^{(-1)}(v) * \mathcal{F}(v)$$

$$= \left[\frac{v}{(1-v)^{n+1}}\right]^{(-1)} * \mathcal{F}(v). \qquad (6)$$

The operator $\mathcal{I}_n$ is called the Noor Integral of n −th order of $\mathcal{F}(v)$. We note that $\mathcal{I}_0\mathcal{F}(v)$=$v\mathcal{F}'(v)$ and $\mathcal{I}_1\mathcal{F}(v)$=$\mathcal{F}(v)$.

Furthermore, it is easily observed that

$$(n+1)\mathcal{I}_n\mathcal{F}(v) - n\mathcal{I}_{n+1}\mathcal{F}(v) = v(\mathcal{I}_{n+1}\mathcal{F}(v))'. \qquad (7)$$

By using hypergeometric functions ${}_2F_1$, (6) becomes

$$\mathcal{I}_n\mathcal{F}(v) = [v\,{}_2F_1(1,1; n+1, v)] * \mathcal{F}(v), \qquad (8)$$

where ${}_2F_1(a, b; c, v)$ is defined by

$${}_2F_1(a, b; c, v) = 1 + \frac{ab}{c}\frac{v}{1!} \qquad (9)$$

$$+ \frac{a(a+1)b(b+1)}{c(c+1)}\frac{v^2}{2!} + \ldots. \qquad (9)$$

For complex parameters

$$\alpha_1,\ldots,\alpha_q\left(\frac{\alpha_i}{A_i}\neq 0,-1,-2,..;i=1,..,q\right)$$

and

$$\beta_1,\ldots,\beta_p\left(\frac{\beta_i}{B_i}\neq 0,-1,-2,..;i=1,..,p\right).$$

In [4] and [23, 24] defined the Fox-Wright generalization ${}_q\Psi_p[\upsilon]$ of the hypergeometric ${}_qF_p$ functions as follows:

$$ {}_q\Psi_p\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_q,A_q);\\ (\beta_1,B_1),\ldots,(\beta_p,B_p);\end{matrix}\upsilon\right] ={}_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right]$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+nA_1)\ldots\Gamma(\alpha_q+nA_q)}{\Gamma(\beta_1+nB_1)\ldots\Gamma(\beta_p+nB_p)}\frac{\upsilon^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(\alpha_i+nA_i)}{\prod_{i=1}^{p}\Gamma(\beta_i+nB_i)}\frac{\upsilon^n}{n!},\qquad(10)$$

where $\alpha_i,\beta_i\in\mathbb{C},\ A_i\in\mathbb{R}^+\ (i=1,\ldots,q)$, $B_i\in\mathbb{R}^+\ (i=1,\ldots,p)$ and $\sum_{j=1}^{p}B_j-\sum_{j=1}^{q}A_j>-1$ for suitable values $|z|$.

For special case, when $A_i=1$ for $i=1,\ldots,q$, and $B_i=1$ for $i=1,\ldots,p$, we have the following relationship:

$${}_qF_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_p;\upsilon)=\Omega\ {}_q\Psi_p\left[(\alpha_i,1)_{1,q};(\beta_i,1)_{1,p};\upsilon\right],\qquad(11)$$

$N\cup\{0\},q\leq p+1;\quad q,p\in N_0=\upsilon\in\mathcal{X},$ where

$$\Omega:=\frac{\Gamma(\beta_1)\ldots\Gamma(\beta_p)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_q)}.\qquad(12)$$

Now, we introduce a function $(\upsilon_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right])^{-1}$ which given by

$$(\upsilon_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right])*(\upsilon_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right])^{-1}$$

$$=\frac{\upsilon}{(1-\upsilon)^{\lambda+1}}=\upsilon+\sum_{n=2}^{\infty}\frac{(\lambda+1)_{n-1}}{(n-1)!}\upsilon^n,\qquad(13)$$

$(\lambda>-1)$,

and obtain the following linear operator:

$$\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)=(\upsilon_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right])^{-1}*\mathcal{F}(\upsilon),\qquad(14)$$

where $\mathcal{F}(\upsilon)\in\mathcal{A},\upsilon\in\mathcal{X},$ and

$$(\upsilon_q\Psi_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p};\upsilon\right])^{-1}=\upsilon+\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\beta_i+(n-1)B_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i+(n-1)A_i)}(\lambda+1)_{n-1}\upsilon^n.$$
(15)

For some computation, we have

$$\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)=\upsilon+\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\beta_i+(n-1)B_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i+(n-1)A_i)}(\lambda+1)_{n-1}u_n\upsilon^n,\qquad(16)$$

where $(u)_n$ denotes the Pochhammer symbol given, in general, by

$$(u)_n=\frac{\Gamma(u+n)}{\Gamma(u)}=\begin{cases}1, & n=0\\ u(u+1)\ldots(u+n-1), & n=\{1,2,\ldots\}.\end{cases}\qquad(17)$$

Special cases:

(i) $\mathcal{J}_0\left[(1,1)_{1,1};(1,1/(n-1))_{1,p}\right]\mathcal{F}(\upsilon)=\mathcal{F}(\upsilon)$

(ii) $\mathcal{J}_1\left[(1,1)_{1,1};(1,1/(n-1))_{1,p}\right]\mathcal{F}(\upsilon)=z\mathcal{F}'(\upsilon)$.

(iii) $\upsilon\left[\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)\right]'$
$=(\lambda+1)\mathcal{J}_{\lambda+1}\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)$
$-\lambda\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)$.

Definition 1 *[7] Let $\mathcal{F}(\upsilon)\in\mathcal{A}$ then $\mathcal{F}(\upsilon)\in\mathcal{S}_\lambda^\mu\left[\mathcal{J}_\lambda(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]$ if and only if*

$$\Re\left\{\frac{\upsilon\left[\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)\right]'}{\mathcal{J}_\lambda\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,p}\right]\mathcal{F}(\upsilon)}\right\}>\mu,0\leq\mu<1,\ \upsilon\in\mathcal{X}.\qquad(18)$$

Noonan and Thomas [15] stated the $q-$th Hankel determinant of the functions $\mathcal{F}(\upsilon)$ of the form (1) for $q\geq 1$ and $n\geq 0$ as

$$H_q(n)=\begin{vmatrix}u_n & u_{n+1} & \cdots & u_{n+q+1}\\ u_{n+1} & u_{n+2} & \cdots & u_{n+q+2}\\ \vdots & \vdots & \ddots & \vdots\\ u_{n+q-1} & u_{n+q} & \cdots & u_{n+2q-2}\end{vmatrix}.$$

(19)

The determinant $H_q(n)$ has been investigated by many authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n\to\infty$ for the functions $\mathcal{F}(\upsilon)$ given by (1) with bounded boundary, Ehrenborg [3] studied the Hankel determinant of exponential polynomials. The Hankel convert of an integer sequence and some properties were discussed by Layman [11].

A classical theorem of Fekete and Szegö functional which considered the Hankel determinante of $\mathcal{F}(\upsilon) \in \mathcal{A}$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} u_1 & u_2 \\ u_2 & u_3 \end{vmatrix}. \qquad (20)$$

It is also known that Fekete Szegö gave sharp estimates of $|u_3 - \mu u_2^2|$ for μ real and $\mathcal{F}(\upsilon) \in \mathcal{A}$. The Fekete-Szegö functional has $|u_3 - \mu u_2^2|$ since received magnificent concern of many authors, specially in connection with several subclasses of the class $\mathcal{A}$ of normalized analytic and univalent functions (see, for example, [1], [14], [19], [21], [22] and [25]).

In this paper, we use the second Hankel determinante ($q = 2,\ n = 2$) for $\mathcal{F}(\upsilon) \in \mathcal{A}$

$$H_2(2) = \begin{vmatrix} u_2 & u_3 \\ u_3 & u_4 \end{vmatrix}.$$

The functional $|u_2u_4 - u_3^2|$ has been studied by several authors as Janteng, Halim and Darus [10] found a sharp bound for $|u_2u_4 - u_3^2|$ where $\mathcal{F}(\upsilon)$ is the subclass RT of , consisting of functions whose derivative has a positive real part. Also see ([8],[9]).

Moreover, Let $\mathcal{P}$ denote the class of functions $\mathrm{p}(\upsilon)$ of the form

$$\mathrm{p}(\upsilon) = 1 + \sum_{n=1}^{\infty} c_n \upsilon^n$$

which are analytic in $\mathcal{X}$ and satisfy

$$\Re\ \mathrm{p}(\upsilon) > 0 \qquad (\upsilon \in \mathcal{X}).$$

Then we say that $\mathrm{p}(\upsilon) \in \mathcal{P}$ is the Carathéodory functions (see [2]).

We first state some preparatory lemmas, needed for proving our proof.

2.Preliminary results

The following lemma can be found in [2] or [20].

Lemma 2 *If a function* $p(\upsilon) = 1 + \sum_{n=1}^{\infty} c_n \upsilon^n \in \mathcal{P}$, *then*

$$|c_n| \le 2 \quad (n = 1,2,3,\dots).$$

The result is sharp for

$$\mathrm{p}(\upsilon) = \frac{1+\upsilon}{1-\upsilon} = 1 + \sum_{n=1}^{\infty} 2\upsilon^n.$$

Using the above lemma, we derive

Lemma 3 *If a function* $p(\upsilon) = 1 + \sum_{n=1}^{\infty} c_n \upsilon^n$ *satisfies the following inequality*

$$\Re\ \mathrm{p}(\upsilon) > \mu \qquad (\upsilon \in \mathcal{X})$$

for some $\mu (0 \le \mu < 1)$, then

$$|c_n| \le 2(1-\mu) \qquad (n = 1,2,3,\dots).$$

The result is sharp for

$$\mathrm{p}(\upsilon) = \frac{1 + (1-2\mu)\upsilon}{1-\upsilon} = 1 + \sum_{n=1}^{\infty} 2(1-\mu)\upsilon^n.$$

Proof. Let

$$q(\upsilon) = \frac{\mathrm{p}(\upsilon) - \mu}{1-\mu} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{1-\mu}\upsilon^n.$$

Note that $\mathrm{q}(\upsilon) \in \mathcal{P}$ and by using Lemma 2, we have

$$\left|\frac{c_n}{1-\mu}\right| < 2 \qquad (n = 1,2,3,\dots)$$

which implies

$$|c_n| \le 2(1-\mu) \qquad (n = 1,2,3,\dots).$$

Lemma 4 *The power series for* $p(\upsilon) = 1 + \sum_{n=1}^{\infty} c_n \upsilon^n$ *converges in* $\mathcal{X}$ *to a function in* $\mathcal{P}$ *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix},$$

$(n = 1,2,3,\dots)$, where $c_{-n} = \overline{c_n}$, are all non-negative. They are strictly positive except for $\mathrm{p}(\upsilon) = \sum_{n=1}^{m} \rho_n \mathrm{p}_0(e^{it_k}\upsilon),\ \rho_n > 0, t_n \ne t_j$ for $n \ne j$; in this case $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \ge m$.

This indispensable and sufficient condition is due to Carathéodory and Toeplitz, and it can be found in [6]. Libera and Złotkiewicz [12],[13] have given the next result by using this lemma with $n = 2,3$.

Lemma 5 *If a function* $p(\upsilon) \in \mathcal{P}$, *then the representations*

$$\begin{cases} 2c_2 = c_1^2 + (4-c_1^2)\zeta \\ 4c_3 = c_1^3 + 2(4-c_1^2)c_1\zeta - (4-c_1^2)c_1\zeta^2 \\ +2(4-c_1^2)(1-|\zeta|^2)\eta \end{cases} \quad (21)$$

for some complex numbers ζ and η ($|\zeta| \le 1, |\eta| \le 1$), are obtained.

By virtue of Lemma 5, we have

Lemma 6 *If a function* $p(\upsilon) = 1 + \sum_{n=1}^{\infty} c_n \upsilon^n$ *satisfies* $\Re\ p(\upsilon) > \mu$ $(\upsilon \in \mathcal{X})$ *for some* $\mu (0 \le \mu < 1)$, *then*

$$2(1-\mu)c_2 = c_1^2 + \{4(1-\mu)^2 - c_1^2\}\zeta$$
$$4(1-\mu)^2 c_3 = c_1^3 + 2\{4(1-\mu)^2 - c_1^2\}c_1\zeta$$
$$-\{4(1-\mu)^2 - c_1^2\}c_1\zeta^2 + 2(1-\mu)$$
$$\{4(1-\mu)^2 - c_1^2\}(1-|\zeta|^2)\eta \quad (22)$$

for some complex numbers ζ and $\eta (|\zeta| \le 1, |\eta| \le 1)$.

Proof. Since

$$q(\upsilon) = \frac{\mathrm{p}(\upsilon) - \mu}{1-\mu} = 1 + \sum_{n=1}^{\infty} \frac{c_n}{1-\mu} \upsilon^n \in \mathcal{P},$$

replacing c_2 and c_3 by $\frac{c_2}{1-\mu}$ and $\frac{c_3}{1-\mu}$ in Lemma 5, respectivily, we immediately have the relations of the lemma.

3 Main Results

Theorem 7 *Let* $\mathcal{F}(\upsilon) \in \mathcal{S}_\lambda^\mu [\mathcal{J}_\lambda(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,p}]$. Then

$$|u_2 u_4 - u_3^2| \le \frac{2(1-\lambda)^2}{\lambda(\lambda+1)^2} \quad (23)$$

$$\left(\frac{\Pi_{i=1}^q \Gamma(\alpha_i + 2A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + 2B_i)}\right)^2. (23)$$

Proof. Since $\mathcal{F}(\upsilon) \in \mathcal{S}_\lambda^\mu [\mathcal{J}_\lambda(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,p}]$, it follows from (18) that

$$\frac{\upsilon [\mathcal{J}_\lambda [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,p}] \mathcal{F}(\upsilon)]'}{\mathcal{J}_\lambda [(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,p}] \mathcal{F}(\upsilon)}$$

$$= \mathrm{p}(\upsilon) \quad (24)$$

for some $\upsilon \in \mathcal{X}$. Equating coefficients in (24) yields

$$u_2 = \frac{c_1}{(\lambda+1)} \frac{\Pi_{i=1}^q \Gamma(\alpha_i + A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + B_i)}$$

$$u_3 = \frac{(c_2 + c_1^2))}{2\lambda(\lambda+1)} \frac{\Pi_{i=1}^q \Gamma(\alpha_i + 2A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + 2B_i)} \quad (25)$$

$$u_4 = \frac{(c_3 + \frac{3}{2} c_1 c_2 + \frac{c_1^3}{2})}{3\lambda(\lambda+1)(\lambda+2)} \frac{\Pi_{i=1}^q \Gamma(\alpha_i + 3A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + 3B_i)}.$$

From (25), it easily established that

$$|u_2 u_4 - u_3^2| = \frac{1}{\lambda(\lambda+1)^2(\lambda+2)}$$
$$\frac{\Pi_{i=1}^q \Gamma[(\alpha_i + A_i)\Gamma(\alpha_i + 3A_i)]}{\Pi_{i=1}^p \Gamma[(\beta_i + B_i)\Gamma(\beta_i + 3B_i)]} \bigg| \frac{c_1 c_3}{3}$$
$$+ \frac{c_1^2 c_2}{2} + \frac{c_1^4}{6} - \frac{(c_2^2 + 2c_2 c_1^2 + c_1^4)(\lambda+2)}{4\lambda}$$
$$\frac{\Pi_{i=1}^p [\Gamma(\beta_i + B_i)\Gamma(\beta_i + 3B_i)]}{\Pi_{i=1}^q \Gamma[(\alpha_i + A_i)\Gamma(\alpha_i + 3A_i)]}$$
$$\left(\frac{\Pi_{i=1}^q \Gamma(\alpha_i + 2A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + 2B_i)}\right)^2 \bigg|$$

or

$$|u_2 u_4 - u_3^2| = H(\alpha_i, A_i, \beta_i, B_i, \lambda) \bigg| \frac{c_1 c_3}{3}$$
$$+ \frac{c_1^2 c_2}{2} + \frac{c_1^4}{6} - \frac{(c_2^2 + 2c_2 c_1^2 + c_1^4)}{4} G \bigg|$$

where

$$H(\alpha_i, A_i, \beta_i, B_i, \lambda) = \frac{1}{\lambda(\lambda+1)^2(\lambda+2)}$$
$$\frac{\Pi_{i=1}^q \Gamma[(\alpha_i + A_i)\Gamma(\alpha_i + 3A_i)]}{\Pi_{i=1}^p \Gamma[(\beta_i + B_i)\Gamma(\beta_i + 3B_i)]}$$

and

$$G = \frac{(\lambda+2)}{\lambda} \frac{\Pi_{i=1}^p [\Gamma(\beta_i + B_i)\Gamma(\beta_i + 3B_i)]}{\Pi_{i=1}^q \Gamma[(\alpha_i + A_i)\Gamma(\alpha_i + 3A_i)]}$$
$$\left(\frac{\Pi_{i=1}^q \Gamma(\alpha_i + 2A_i)}{\Pi_{i=1}^p \Gamma(\beta_i + 2B_i)}\right)^2$$

Since the function $\mathrm{p}(\upsilon)$ is a member of the class $\mathcal{P}$ at the same time, we can suppose without restriction that $c_1 \ge 0$ and take

$c_1 = c$ $(0 \le c \le 2)$ and by using (22)we derive

$$|u_2 u_4 - u_3^2| = H(\alpha_i, A_i, \beta_i, B_i, \lambda)$$
$$\bigg| \frac{1}{12(1-\mu)} [c^4 + 2c^2\{4(1-\mu)^2 - c^2\}\zeta$$
$$-c^2\{4(1-\mu)^2 - c^2\}\zeta^2 + 2(1-\mu)c$$
$$\{4(1-\mu)^2 - c^2\}(1-|\zeta|^2)\eta] + \frac{1}{4(1-\mu)}$$

$$[c^4 + c^2\{4(1-\mu)^2 - c^2\}\zeta] \ + \frac{c^4}{6}$$

$$-\frac{G}{16(1-\mu)^2}[c^4 \ + 2c^2\{4(1-\mu)^2$$

$$- c^2\}^2\zeta^2] \ - \frac{G}{4(1-\mu)}$$

$$[c^4 + c^2\{4(1-\mu) - c^2\}\zeta] - \frac{c^4}{4}G\Big|$$

An application of triangle inequality and alteration of $|\zeta|$ by ρ give

$$|u_2u_4 - u_3^2| \leq H(\alpha_i, A_i, \beta_i, B_i, \lambda)$$

$$\left[\left\{\frac{1}{3(1-\mu)} - \frac{(4\mu^2 - 12\mu + 9)}{16(1-\mu)^2}G\right\}c^4\right.$$

$$+\left\{\frac{5}{12(1-\mu)} - \frac{(3-2\mu)}{8(1-\mu)^2}G\right\}$$

$$c^2\{4(1-\mu)^2 - c^2\}\rho + \left\{\frac{c^4}{12(1-\mu)}\right.$$

$$\left.+\frac{\{4(1-\mu)^2 - c^2\}}{16(1-\mu)^2}G\right\} - \frac{c}{6}\{4(1-\mu)^2$$

$$\left.-c^2\}\rho^2 + \tfrac{1}{6}c\{4(1-\mu)^2 - c^2\}\right] \quad = F(c,\rho),$$

(26)

where $0 \leq c \leq 2$ and $0 \leq \rho \leq 1$.

We next maximize the function $F(c,\rho)$ on the closed square $[0,2] \times [0,1]$.

Since

$$\frac{\delta F}{\delta \rho} = H(\alpha_i, A_i, \beta_i, B_i, \lambda)$$

$$\left[\left\{\frac{5}{12(1-\mu)} - \frac{(3-2\mu)}{8(1-\mu)^2}G\right\}\right.$$

$$c^2\{4(1-\mu)^2 - c^2\} + \left\{\frac{c^4}{6(1-\mu)}\right.$$

$$\left.+\frac{\{4(1-\mu)^2 - c^2\}}{8(1-\mu)^2}G - \frac{c}{3}\right\}$$

$$\{4(1-\mu)^2 - c^2\}\rho],$$

$$\frac{(3-2\mu)}{(1-\mu)}G < \frac{10}{3}, \quad \text{wehave} \quad \frac{\delta F}{\delta \rho} > 0.$$

Thus $F(c,\rho)$ cannot have a maximum in the interior of the closed square $[0,2] \times [0,1]$. Then, for fixed $c \in [0,2]$

$$\max_{0 \leq \rho \leq 1} F(c,\rho) = F(c,1) = M(c)$$

$M'(c) < 0$ for $0 < c < 2$ and has real critical point at $c = 0$. Also observe that $M(c) > M(2)$. Therefore, $\max_{0 \leq c \leq 2} M(c)$ occurs at $c = 0$ and thus the upper bound of (26) corresponds to $\rho = 1$ and $c = 0$, in which case

$$|u_2u_4 - u_3^2| \leq \frac{2(1-\lambda)^2}{\lambda(\lambda+1)^2}$$

$$\left(\frac{\Pi_{i=1}^{q}\Gamma(\alpha_i + 2A_i)}{\Pi_{i=1}^{p}\Gamma(\beta_i + 2B_i)}\right)^2. \qquad (27)$$

This complete the proof.

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