Mansoura University
Faculty of Engineering
Math. & Eng. Physics. Dept.

Mathematics 3

1^{st} year
Final Exam, Jan. 2011
Time Allowed: 3 Hours.

يتألف الإختبار من ٤ أسئلة في صفحتين. برجاء بدء إجابة كل فرع من إحدى نهايتي ورقة الإجابة.

Question (1)

(a) [25 marks] Solve the following differential equations **using any method**

i. $\frac{di(t)}{dt}+i(t)=t, \quad i(0)=c$

ii. $(4xy+3y^2-x)dx+x(x+2y)dy=0$

iii. $(x^2-xy+y^2)dx-xy\,dy=0$

iv. $y''-4y'+3y=10e^t, \quad y(0)=0, \; y'(0)=-3$

v. $(D^3+6D^2+12D+8)y=e^{-2t}, \quad y(0)=1, y'(0)=-2, y''(0)=4$

(b) [5 marks] Solve the following system

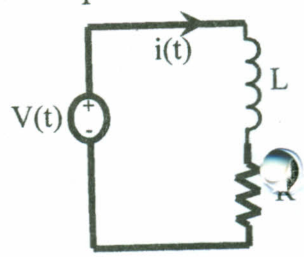
$$x'-x+5y'=-1, \quad y'-4y-2x=-2, \text{ where } \quad x(0)=y(0)=0$$

Question (2)

(a) [12 marks] Evaluate

i. $\int_0^\infty e^{-t}\frac{(\cos t-1)}{t}dt$ ii. $L^{-1}\left[e^{-3s}\frac{s}{(s^2+1)^2}\right]$ iii. $D^\alpha t^3, \quad 0<\alpha<1$

(b) [12 marks] If the voltage drop across the inductor is modeled by $LD^\alpha i(t)$, $0<\alpha\le 1$ where L is the inductance, D^α is Caputo fractional derivative operator.



i. Prove that the current across the R-L circuit is given by

$$i(t)=ct^{\alpha-1}E_{\alpha,\alpha}\left(-t^\alpha\right)+t^{\alpha+1}E_{\alpha,\alpha+2}\left(-t^\alpha\right). \quad 0<\alpha\le 1$$

$$[L=1\,(H),\; R=1\,(\Omega),\; i(0)=c, \text{ and } V(t)=t]$$

ii.Obtain $i(t)$ for $\alpha=1$, **and compare with the solution of problem (1-a-i)**

(c) [6 marks] Explain the difference between $D^1y(t_0)$ and $D^\alpha y(t_0)$, $0<\alpha<1$

Note: you may to need these relations through the exam

$$L^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha+a}\right]=t^{\beta-1}E_{\alpha,\beta}\left(-at^\alpha\right), \quad z^2E_{1,3}(z)=e^z-z-1, \quad (s+2)^3=s^3+6s^2+12s+8$$

3. Given the function $\varphi(x,y) = e^{-x^2y+1}$ and the point $p = (1,1)$.

(a) [5 pts] Given that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Find $I = \int_0^\infty \varphi(x,y)dx$.

(b) [5 pts] Expand $\varphi(x,y)$ in a Taylor series about the point p.

(c) [5 pts] Let S be the surface $g(x,y,z) = 0$, where $g(x,y,z) = \varphi(x,y) - z$, and let $q = (1,1,z_0)$ be a point on S. Use the results obtained in part (b) to get the following at the point q:

- **i.** The equation of the normal line to S.
- **ii.** The equation of the Tangent plane to S.
- **iii.** The gradient of $g(x,y,z)$.
- **iv.** The maximum rate of change of $g(x,y,z)$ and its direction.
- **v.** $\frac{\partial y}{\partial x}$ by applying the implicit differentiation rule to $g(x,y,z) = 0$.

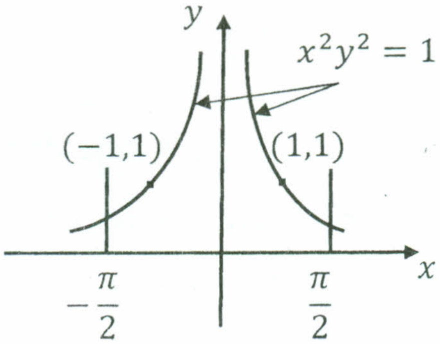
(d) [5 pts] Let $x = r\cos\theta$, $y = r\sin\theta$. Applying the chain rule to evaluate φ_r and φ_θ, substitute with x and y to write your results in terms of r and θ. Finally show that $\varphi_{rr} = \varphi_{xx} + \varphi_{yy} + 2\varphi_{xy}x_r y_r$.

(e) [5 pts] Evaluate $\int_0^1 \int_{1/\sqrt{y}}^\infty \varphi(x,y)dxdy + \int_1^\infty \int_1^\infty \varphi(x,y)dxdy$.

(g) [5 pts] Use Euler's theorem to evaluate

- **i.** $x\varphi_x + y\varphi_y$,
- **ii.** $x^2\varphi_{xx} + 2xy\varphi_{xy} + y^2\varphi_{yy}$.

(f) [7 pts] Find the extreme values of $h(x,y) = \varphi(x,y)\sin x$ on the region $R: y \geq 0, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x^2y^2 \leq 1$.



4. Consider the vector field $F = 2y(z+4)\,i + j + (xy+3z)\,k$ and the volume $R: x^2 + y^2 + (z+4)^2 \leq 25, z \geq 0$.

(a) [5 pts] Verify Green's theorem for the vector field F and the lower surface of R.

(b) [7 pts] Verify Stokes' theorem for the vector field F and the upper surface of R.

(c) [11 pts] Verify Gauss' theorem for the vector field F and the volume R.

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