NUMERICAL METHOD FOR THE SOLUTION OF NONLINEAR BOUNDARY-VALUE PROBLEMS

Hamza M. Habib
Department of Basic Engineering Sciences
El-Menoufia University
Shebin El-Kom, Egypt

Mohamed Magdy M. El-Farrash
Department of Basic Engineering Sciences
El-Menoufia University
Shebin El-Kom, Egypt

Mohamady A. Basiouny
Department of Basic Engineering Sciences
El-Menoufia University
Shebin El-Kom, Egypt

ABSTRACT

A simple, efficient numerical solution method for nonlinear boundary value problems is presented. The approach depends on linearizing the finite-differenced equations for the dependent variables then solving the correction equation for the dependent variable by the tri-diagonal matrix method, finally correcting the solution at each iteration. The method is somewhat similar to Newton's method in that the Jacobian is evaluated at each iteration,

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but does not require the evaluation of the inverse of the Jacobi matrix at each iteration which means saving both computing time and memory usage. Two test problems were chosen to check the validity of the technique.
The proposed method is very efficient if applied to boundary-value ordinary differential equation or one-dimensional time dependent partial differential equation.

KEYWORDS

Numerical, Boundary Value, Nonlinear, Differential

INTRODUCTION

Numerical methods are useful for solving fluid dynamics, heat and mass transfer problems, and other partial differential equations of mathematical physics when such problems cannot be handled by the exact analysis techniques because of nonlinearities, complex geometries and complicated boundary conditions, one generally resorts to numerical technique for solution. The boundary value problems become nonlinear due to the nonlinearity of the governing equations, or of the boundary conditions or both. Most physical problems are actually nonlinear. The nonlinearity of these equations reflects the main features of the system. Therefore, the nonlinear terms should not be replaced by a linear term because the main features may be lost during the linearization process. When finite-difference approach is used, the problem domain is discretized so that the values of the unknown dependent variables are considered only at a finite number of nodal points instead of every point over the region.

There is no difficulty in applying the finite-difference approximation to discretize a nonlinear problem; but the difficulty is associated with the solution of the resulting system of algebraic equations.

Several references on the fundamentals of discretization and finite-difference method include Richtmeyer and Morton (1967), Smith (1978), Andreson (1978), Tannchill and Pletcher (1984), Berezin and Zhidkov (1965) and Roache (1976).

In this paper a new technique was developed for solving nonlinear boundary value problems by iterations.

OBJECTIVES

The objective of the work reported here was to develop a simple and efficient numerical method for the solution of nonlinear boundary value problems with the following features.

1. The method should be simple and have good convergence characteristics,
2. Evaluation of the inverse should not be required

In the following section the method will be explained through two examples.

Test problem 1:

Application to Ordinary Differential Equation

Consider the following nonlinear boundary value Problem

$$y''(x) = \frac{2}{[y'(x) + 1]} \qquad 1 < x < 4 \qquad (1)$$

with the boundary conditions

$$y(1) = 1/3 \qquad y(4) = 20/3$$

The exact solution for this problem is

$$y(x) = \frac{4}{3} x^{3/2} - x \qquad (2)$$

The differential equation (1) is discretized by the central difference scheme. The domain a < x < b is divided into M equal subregions each of thickness $\Delta x = (b - a)/M$ as illustrated in Fig. 1.

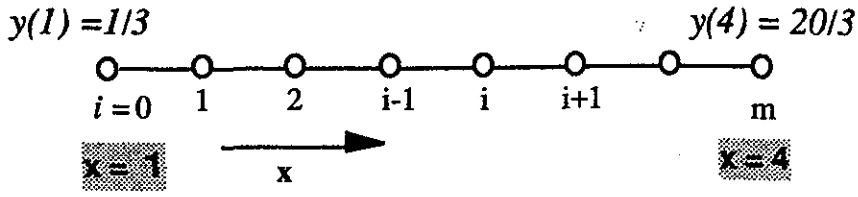

Fig 1. Computational Grid for Problem 1

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} + \frac{y_{i+1} - y_{i-1}}{2\,\Delta x}\,\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} - 2 = 0 \quad (3)$$

$$y_{i+1}^2 - 2\,y_i\,y_{i+1} + 2\,y_i\,y_{i-1} - y_{i-1}^2 + 2\,\Delta x\,y_{i+1}$$
$$- 4\,\Delta x\,y_i + 2\,\Delta x\,y_{i-1} - 4\,(\Delta x)^3 = 0$$

which can be written as

$$f(y_{i-1}, y_i, y_{i+1}) = y_{i+1}^2 - 2\,y_i\,y_{i+1} + 2\,y_i\,y_{i-1} - y_{i-1}^2 + 2\,\Delta x\,y_{i+1}$$
$$- 4\,\Delta x\,y_i + 2\,\Delta x\,y_{i-1} - 4\,(\Delta x)^3 \quad (4)$$

Equation (4) gives a set of nonlinear algebraic equations for the y_i. Methods of solution of such systems can be found in Rheinboldt (1974), Ortega and Rheinboldt (1970), Traub (1964)

By linearizing equation (4) using the first two terms of Taylor's series

$$\{\, y_{i+1}^2 - 2\,y_i\,y_{i+1} + 2\,y_i\,y_{i-1} - y_{i-1}^2 + 2\Delta x\,y_{i+1} - 4\Delta x\,y_i + 2\Delta x\,y_{i-1}$$
$$- 4(\Delta x)^3 \,\}_k + \left\{\frac{\partial f_i}{\partial y_{i-1}}\right\}_k \varepsilon_{i-1} + \left\{\frac{\partial f_i}{\partial y_i}\right\}_k \varepsilon_i + \left\{\frac{\partial f_i}{\partial y_{i+1}}\right\}_k \varepsilon_{i+1} = 0$$

or

$$\left\{\frac{\partial f_i}{\partial y_{i-1}}\right\}_k \varepsilon_{i-1} + \left\{\frac{\partial f_i}{\partial y_i}\right\}_k \varepsilon_i + \left\{\frac{\partial f_i}{\partial y_{i+1}}\right\}_k \varepsilon_{i+1} = -\{\, y_{i+1}^2 - 2\, y_i\, y_{i+1}$$

$$+ 2\, y_i\, y_{i-1} - y_{i-1}^2 + 2\Delta x\, y_{i+1} - 4\Delta x\, y_i + 2\Delta x\, y_{i-1} - 4(\Delta x)^3 \,\}_k \tag{5}$$

where

$$\varepsilon_i = y_i^{(k+1)} - y_i^{(k)} \tag{6}$$

and

$$\left\{\frac{\partial f_i}{\partial y_{i-1}}\right\}_k = 2\, y_i - 2\, y_{i-1} + 2\, \Delta x$$

$$\left\{\frac{\partial f_i}{\partial y_i}\right\}_k = -2\, y_{i+1} + 2\, y_{i-1} - 4\, \Delta x$$

$$\left\{\frac{\partial f_i}{\partial y_{i+1}}\right\}_k = 2\, y_{i+1} - 2\, y_i + 2\, \Delta x$$

Equation (5) gives a set of linear algebraic equations for the corrections ε_i with a matrix of coefficients of tri-diagonal form which can be solved by Thomas algorithm (Patankar, 1980). Once the corrections are calculated for this level of iterations. The corresponding y_i can be calculated from.

$$y_i^{(k+1)} = y_i^{(k)} + \varepsilon_i \tag{7}$$

x	Numerical Solution	Exact Solution	Error
2	1.775	1.771	3.312E-03
3	3.930	3.928	2.218E-03

Table 1. Comparison of the Exact Solution with the proposed method for very coarse grid (2 interior nodal points)

Table (1) shows a comparison of the proposed numerical method with the exact solution for a very coarse grid 4 grid points (2 interior grid points).

In figure (2) the numerical solution is compared to the analytical solution. for a finer grid (20 interior grid points). The numerical solution was obtained after 15 iterations for the accuracy. The comparison shows a good agreement between the numerical and analytical method.

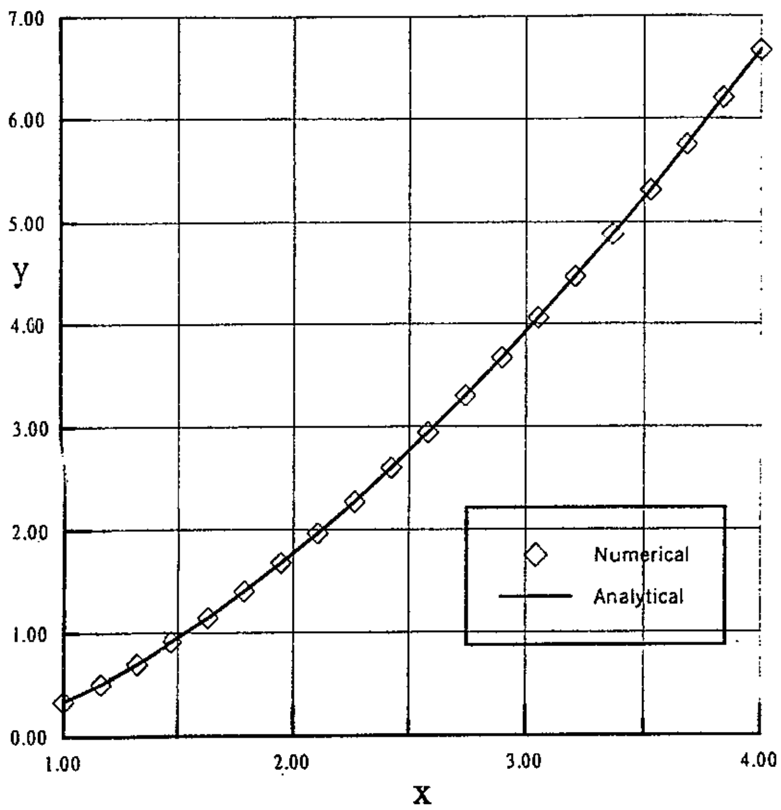

Fig 2. Comparison between Numerical and Analytical Solutions

Test Problem 2

Application to One-dimensional time-dependent Nonlinear Partial Differential Equation.

Consider a nonlinear differential equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} \tag{8}$$

Where m is a positive integer and m $\geq$ 2. This equation can be written in the form.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[m\, u^{m-1} \frac{\partial u}{\partial x} \right] \tag{9}$$

The finite difference approximation for equation (8) is

$$\frac{u(i,j+1)-u(i,j)}{\Delta t} = \theta \Delta_{xx} \; u^m(i,j+1) + (1-\theta)\Delta_{xx} \, u^m(i,j) \tag{10}$$

Where

$$\Delta_{xx} f_i = \frac{f_{i-1} - 2 f_i + f_{i+1}}{(\Delta x)^2} \tag{11}$$

is the central difference operator for the second derivative and the weight factor θ ($0 < \theta < 1$) controls the degree of implicitness. The finite difference approximation given by equation (10) is not convenient for computational purposes because the resulting system of algebraic equations is highly nonlinear, hence is difficult to solve. To alleviate this difficulty, the unknown $u^m(i, j+1)$ is linearized by the following procedure discussed by Richtmyer and Morton (1967) and Smith (1984). A Taylor series expansion of $u^m(i, j+1)$ about j gives

$$u^m(i,j+1) = u^m(i,j) + \frac{\partial u^m(i,j)}{\partial t} \Delta t + ...$$

$$u^m(i,j+1) = u^m(i,j) + \frac{\partial u^m(i,j)}{\partial u(i,j)} \frac{\partial u(i,j)}{\partial t} \Delta t + ...$$

$$u^m(i,j+1) = u^m(i,j) + m \; u^{m-1}(i,j) \frac{u(i,j+1) - u(i,j)}{\Delta t} \Delta t + ...$$

$$u^m(i,j+1) = u^m(i,j) + m \; u^{m-1}(i,j) [u(i,j+1) - u(i,j)] + ...$$

A result which replaces the non-linear unknown $u^m(i, j+1)$ by an approximate linear in $u(i, j+1)$

Define $\varepsilon_i = u(i,j+1) - u(i,j)$

Thus equation (10) becomes

$$\frac{\varepsilon_i}{\Delta t} = \theta \Delta_{xx} \left[u^m(i,j) + m\, u^{m-1}(i,j)\, \varepsilon_i \right] + (1-\theta)\Delta_{xx}\, u^m(i,j)$$

$$\frac{\varepsilon_i}{\Delta t} = m\, \theta\, \Delta_{xx}\, u^{m-1}(i,j)\, \varepsilon_i + \Delta_{xx}\, u^m(i,j)$$

$$\frac{\varepsilon_i}{\Delta t} = m\, \theta \left[u^{m-1}(i-1,j)\, \varepsilon_{i-1} - 2\, u^{m-1}(i,j)\, \varepsilon_i \right.$$

$$\left. + u^{m-1}(i+1,j)\, \varepsilon_{i+1} \right] + u^m(i-1,j) - 2\, u^m(i,j) + u^m(i+1,j) \qquad (12)$$

Which gives a set of linear equations for ε_i. The solution at the $(j+1)^{th}$ time-level is obtained from

$$u(i,j+1) = u(i,j) + \varepsilon_i \qquad (13)$$

define $\qquad r = \Delta t/(\Delta x)^2$

thus equation (12) becomes

$$-r\, m\, \theta\, u^{m-1}(i-1,j)\, \varepsilon_{i-1} + \left[1 + 2\, r\, m\, \theta\, u^{m-1}(i,j)\right] \varepsilon_i$$
$$- r\, m\, \theta\, u^{m-1}(i+1,j)\, \varepsilon_{i+1} = r \left[u^m(i-1,j) - 2\, u^m(i,j) \right.$$
$$\left. + u^m(i+1,j) \right] \qquad (14)$$

Equation (14) is unconditionally stable with no restriction on the value of the parameter r. The only restriction on r is that for a given α and Δx, the resulting value of the time step Δt should not be large to impair accuracy. Equation 14 gives a set of linear equations for ε_i.with a matrix of coefficients of tri-diagonal form which can be solved directly at the j^{th} time level for the corrections ε_i by Thomas algorithm. Once the corrections are calculated for this level of iterations the corresponding u(i, j +1) can be calculated from.

$$u(i,j+1) = u(i,j) + \varepsilon_i \qquad (15)$$

Figure 3 illustrates the finite-difference molecules for the Crank-Nicolson implicit scheme.

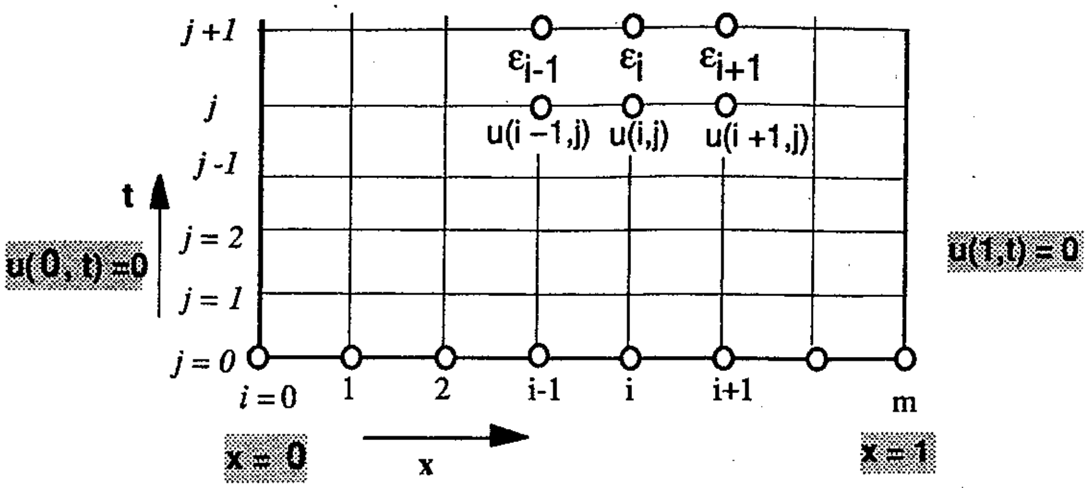

Fig 3. Computational Grid for Problem 2

For the special case m = 2, equation (8) is just like a nonlinear transient heat conduction equation with temperature dependent thermal conductivity. If we choose $\theta = 0.5$ (Crank-Nicolson), $\Delta x = 0.05$, and $\Delta t = 0.025$ thus r = 10

Figure 4 shows the temperature distribution for different time level.

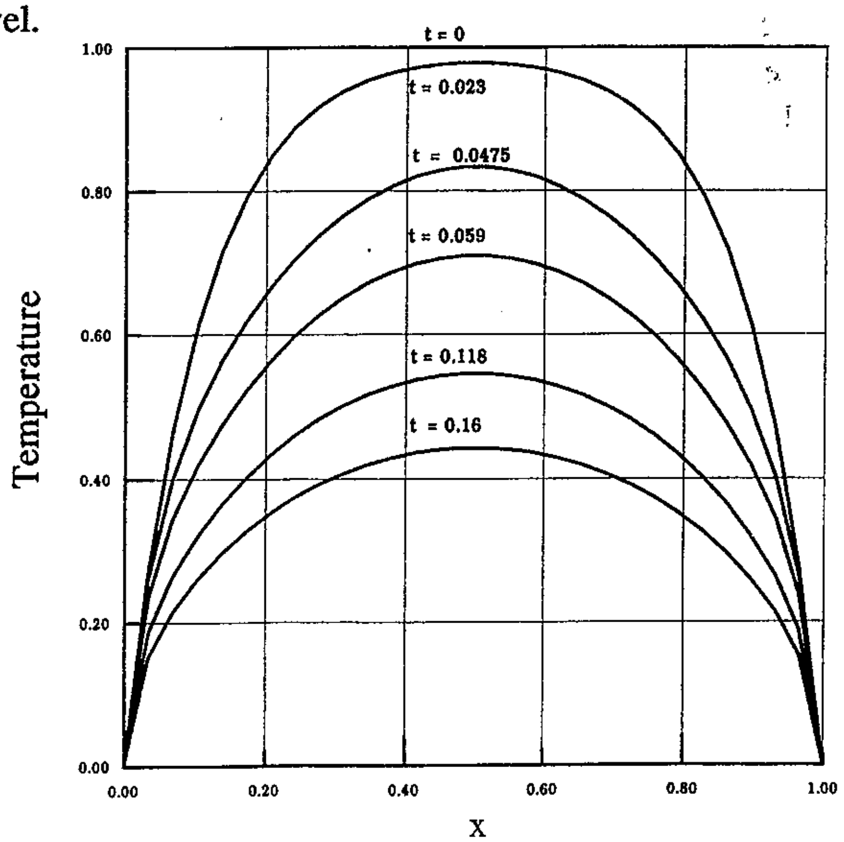

Fig 4. Temperature Distribution along a thin rod at different times

A comparison was not made since an exact solution was not available but the judgement on the result can be guessed from the convergence rate and the steady state solution which is eventually zero.

Summary of the Numerical Method

1- Linearize the discretiztion equations with Taylor's series
2- Guess the solution u*
3- Solve the correction equation ε_i
4- Calculate u by adding the correction, ε_i to the guessed value u*.
5- Treat the corrected u as a new guessed value u*, return to step 2, and repeat the whole procedure until a converged solution is obtained.

DISCUSSION

The method is somewhat similar to Newton's method in that the Jacobian is evaluated at each iteration. However, it doesn't require the inversion of the Jacobian at each iteration
The two test problems chosen gave a very high convergence rate even with a very coarse grid. If the differential equation is 2 dimensional or more the convergence rate will be slower than Newton's method but it is more compatible with the nature of the nonlinear problems which could not be solved directly. But we will obtain more efficient and economical solution by our proposed method.

The method of course, is not free from difficulties. For instance, it does not always converge to the solution; however, this is typical of all numerical techniques for nonlinear equation systems. The failure of the method to converge is typically due. This difficulty could be circumvented by choosing better initial values of the variables. However, in many cases this may not be feasible.

CONCLUSIONS

A simple, efficient numerical solution method for nonlinear boundary- value problems is presented. The method gave the same convergence rate as Newton's method if applied to ordinary differential equation or one-dimensional, time-dependent partial differential equation with the advantage of no matrix inversion is involved. Based on the successful solution of large number of problems, of which a two have been presented here, we believe that the objectives have been met.

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الطرق العددية لحل المعادلات التفاضلية غير الخطية

ذات الشـــروط الحدية

د./ حمزه محمد حبيب — قسم العلوم الأساسية الهندسية، كلية الهندسة بشبين الكـــوم

د./ محمد مجدى محمد الفراش — قسم العلوم الأساسية الهندسية، كلية الهندسة بشبين الكـــوم

د./ محمدي أبو الفتوح بسيونى — قسم العلوم الأساسية الهندسية، كلية الهندسة بشبين الكـــوم

ملخص البحث باللغة العربية:

- يقدم هذا البحث طريقة مبسطة وذات كفاءة عالية لحل المعادلات التفاضلية غـــير الخطية ذات الشروط الحدية . وتعتمد الطريقة علي جعل معادلات الفروق خطيـــة في متغيرات التصحيح وبعد ذلك تحل معادلات التصحيح مباشرة بطريقة تومـــاس للمصفوفة الثلاثية الأقطار .

- وتتشابه الطريقة مع طريقة نيوتن في حساب الجاكوبيان في كل محاولـــة ولكنـــها تختلف عنها في أننا لا نحتاج الى حساب المصفوفة العكسية للجاكوبيان فـــي كـــل محاولة وبذلك نوفر كل من الوقت والجهد والذاكرة المستخدمة في الحاسب الآلي .

- وقد جربت الطريقة علي عدة مسائل مختلفة وقدم منها مثالين أحدهما علي المعادلات التفاضلية العادية الغير خطية والآخر علي معادلة تفاضلية جزئية غير خطية فـــي متغيرين أحدهما الزمن وقد أثبتت الطريقة نجاحها.