CHARACTERIZATION OF FUZZY TO AND Ro TOPOLOGICAL SPACES

" خصائص الفراغات التوبولوجيه الغازيه من النـــوع Ro,To"

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ABSTRACT

Many authors investigated fuzzy To and fuzzy Ro spaces depending upon the ordinary points of a set and not the fuzzy points. It is the purpose of this note to suggest new definitions of fuzzy To and fuzzy Ro - spaces using Wong definition of fuzzy points. It will be also shown that these new definitions are equivalent to those introduced by Srivastava. Moreover the properties of To - ness and Ro - ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is To or Ro if the original topological space is To or Ro, respectively.

الخـــلاصه :

كثير من الابحاث في هذا المجال بحثت كل من الفراغ الغازى من النوع To ، والفراغ الغازى من النوع Ro,To معتمدين على النقط العاديه لفئه وليست على النقط الغازيه. والغرض من هذا العمل هو اقتراح تعريفـــات جديده للفراغات الغازيه من النوع To ، Ro باستخدام تعريف ونج للنقط الغازيه . وقد اثبتنا ان هـــذه التعريفات الجديده تختلف عن مثيلتها المعرفه في كثير من الابحاث ، حيث انها (اى التعريفات الجديده المقترحه) تعتمد على النقط الغازيه وليست على النقط العاديه . وقد امكن اثبات نظريه تيخنوف لحواصل الضرب اللانهائـــى وكذلك الخاصيه الوراثيه لهذه الفراغات ، بالاضافه الى اثبات انها امتداد جيـــد بمفهوم ليـــون لمثيلتهـــا فـــى الفراغات التوبولوجيه العاديه .

1. INTRODUCTION

The fundamental concept of a fuzzy set was introduced by Zadeh in 1965 [8]. Since then, intensive studies of fuzzy sets have been developed. In particular, the definition of a fuzzy point was first given in 1974 by Wong in [7]. It is notable that, with this definition an ordinary point of a set is not a special case of a fuzzy point. In 1980, Pu and Liu [3] remedy this drawback by redefining fuzzy points in a way that can be used to develop the theory of fuzzy topology in a satisfactory way. In 1984, R. Srivastava, Lal, and A.K. Srivastava [4] studied the concept of a fuzzy T_1-topological space using the Wong fuzzy point [7]. Later in 1988, they introduced an equivalent definition (depending upon the ordinary points of a set) of a fuzzy T_1-space [6]. On the other hand, a fuzzy T_0-topological space has been defined and studied by Hutton and Reilly [1], Pu and Liu [3], and R. Srivastava, Lal, and A.K. Srivastava [5]. Hutton [1] and Srivastava [5] studied, in addition, the concept of a fuzzy R_0-space. It can be seen that in papers [1, 2, 3] the authors investigated fuzzy T_0 and fuzzy R_0 spaces depending upon the ordinary points of a set and not the fuzzy points. It is the purpose of this note to suggest new definitions of fuzzy T_0 and fuzzy R_0-spaces using the Wong definition of fuzzy points [7]. It will be also, shown that these new definitions are equivalent to those introduced by Srivastava in [5]. Moreover, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is T_0 or R_0 if the original topological space is T_0 or R_0, respectively.

2. BASIC DEFINITIONS AND PROPERTIES

A function A from a nonempty set X to the unit interval [0,1] is called a fuzzy set in X. The membership function of a fuzzy set A in X will be denoted by μ_A. A fuzzy topology τ on X is a collection of fuzzy subsets in X which is closed under arbitrary suprema and finite infima, contains both ϕ and X and, in addition, it contains all constant fuzzy sets. The term "fuzzy topological space" will be abbreviated as fts. A fuzzy point in X is a fuzzy set $p : X \to [0,1]$ such that $p(x) = t$ for $x = x_p$, and $p(x) = 0$, otherwise, where $t \in (0,1]$. x_p is called the support of p and t, its value. A fuzzy point p is said to belong to a fuzzy set A in X ($p \in A$) if $p(x_p) < A(x_p)$. If A is a subset of X, we shall denote the characteristic function of A, also, by A.

3. FUZZY T_0-TOPOLOGICAL SPACES

DEFINITION 3.1. (Pu and Liu [3]). An fts (X,τ) is said to be a fuzzy T_0 topological space iff (X,τ) is quasi T_0 and for any $s, t \in [0,1)$ and $x, y \in X$, $x \neq y$ $\exists\, U \in \tau$ such that $U(x) = s$ and $U(y) > t$, or $U(x) > s$ and $U(y) = t$.

DEFINITION 3.2. (Hutton and Reilly [1]). An fts (X,τ) is said to be fuzzy T_0 iff each fuzzy set in X can be written as $\sup_i \inf_j U_{ij}$, where U_{ij}, $i \in I$, $j \in J$, is fuzzy open or fuzzy closed.

DEFINITION 3.3. (Srivastava [5]). An fts (X,τ) is said to be fuzzy T_0 iff $\forall\, x,y \in X$, $x \neq y$, $\exists\, U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$.

Now we introduce our new definition of a fuzzy T_0 -topological space.

DEFINITION 3.4. An fts (X,τ) is said to be fuzzy T_0 iff for any two distinct fuzzy points p, q in X, $\exists\, U \in \tau$ such that $p \in U$ and $q \notin U$ or $q \in U$ and $p \notin U$.

We now compare the above four definitions of fuzzy-T_0-ness in the following theorem.

THEOREM 3.1. Consider the following statements for the fts (X,τ) :

(I) For any distinct fuzzy points p, q in X, $\exists\, U \in \tau$ such that $p \in U$ and $q \notin U$ or $q \in U$ and $p \notin U$.

(II) $\forall\, x,y \in X$, $x \neq y$, $\exists\, U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$.

(III) Each fuzzy set in X can be written in the form $\sup_i \inf_j U_{ij}$, where each U_{ij}, $i \in I$, $j \in J$, is a fuzzy open or a fuzzy closed set.

(IV) (X,τ) is quasi T_0 and, for any two disrtinct points $x, y \in X$ and for all $s, t \in [0,1)$, there exists $U \in \tau$ such that either $U(x) = s$ and $U(y) > t$ or $U(x) > s$ and $U(y) = t$.

We have the following implications :

(I)	$\Leftrightarrow$	(II)
(I)	$\Rightarrow$	(III)
(III)	$\not\Rightarrow$	(I)
(I)	$\Rightarrow$	(IV)
(IV)	$\not\Rightarrow$	(I)

Proof. It sufficies to prove that (I) $\Leftrightarrow$ (II). The remaining implications follows directly using [5, Theorem 2.1].

(I) $\Rightarrow$ (II). Let $x, y \in X$, $x \neq y$ and let p_n, q_n be fuzzy points in X with supports x, y, respectively, and such that $p_n(x) = q_n(y) = 1 - \frac{1}{n}$, $n \in N$. Since $x \neq y$ then $p_n \neq q_n$ for every $n \in N$ and by (I) $\exists\, U_n \in \tau$ such that either $p_n \in U_n$ and $q_n \notin U_n$, or $q_n \in U_n$ and $p_n \notin U_n$. Assume that $p_n \in U_n$ and $q_n \notin U_n$ (the other case can be treated similarly) then $U_n(x) > 1 - \frac{1}{n}$. Define $U = \bigcup_n U_n$ then, $U \in \tau$ and $U(x) = 1$, $U(y) = 0$. So we have (II).

(II) $\Rightarrow$ (I). Suppose that p, q are two distinct fuzzy points in X with supports x,y and values $r, s \in (0,1)$, respectively, then $x \neq y$ and by (II) $\exists\, U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$, or $U(x) = 0$ and $U(y) = 1$. Assume that $U(x) = 1$ and $U(y) = 0$ (the other case can be treated similarly). Since $p(x) = r < 1$, and $q(y) = s > 0$, it follows that $p \in U$ and $q \notin U$. So we have (I).

REMARK 3.1. Definition 3.4 can be replaced by an equivalent definition when we replace the fuzzy open set U by a fuzzy closed set V. In this case all the implications of theorem 3.1 remain valid.

The following theorem shows that the property of T_0-ness of a fuzzy topological space is productive.

THEOREM 3.2. Let $\{(X_i,\tau_i) : i \in I\}$ be a family of fuzzy T_0-topological spaces (in the sense of Definition 3.4), then the product space $(X,\tau) = \prod_i (X_i,\tau_i)$ is a fuzzy T_0 iff each coordinate fts is fuzzy T_0.

Proof. Let (X_j, τ_j) be fuzzy T_0 for $j \in I$ and let p, q be two distinct fuzzy points in X, $p = \langle p_j \rangle$, $q = \langle q_j \rangle$. Then $p_i \neq q_i$ for at least one $i \in I$. Then $\exists\, U_i \in \tau_i$ such that $p_i \in U_i$ and $q_i \notin U_i$ or $q_i \in U_i$ and $p_i \notin U_i$. Suppose that $p_i \in U_i$ and $q_i \notin U_i$ (the other case can be treated similarly). Let $U = \prod_j U'_j$, where $U'_j = X_j$, for $j \neq i$,

$U_j' = U_j$ for $j \neq i$. It is clear that $U \in \tau$ and $p \in U$, $q \notin U$. Hence (X,τ) is fuzzy T_0. Conversely let (X,τ) be fuzzy T_0 and consider any (X_i, τ_i), $i \in I$. Let p_i, q_i be two distinct fuzzy points in X_i and construct the two distinct fuzzy points $p = < p_j' >$, $q = < q_j' >$ in X where $p_j' = q_j'$ for $j \neq i$ and $p_i' = p_i$, $q_i' = q_i$. Then $\exists\, U \in \tau$ such that either $p \in U$ and $q \notin U$, or $q \in U$ and $p \notin U$. Suppose that $p \in U$ and $q \notin U$ (the other case can be treated similarly). Then we can find a basic fuzzy open set $\prod_j U_j$ such that $p \in \prod_j U_j \subset U$. It follows that $p_i \in U_i$, and since $q \notin U$ then $q \notin \prod_j U_j$ and hence $q_i \notin U_i$. This proves that (X_i, τ_i) is fuzzy T_0.

Using the definitions of a fuzzy subspace introduced by Pu and Liu [3, Definition 8.1] and the topologically generated fuzzy topological space (introduced by Lowen [2]) together with Definition 3.4 we can easily prove the follwing theorems.

THEOREM 3.3. Every fuzzy subspace of a fuzzy T_0-space is also a fuzzy T_0-space.

THEOREM 3.4. Let (X, T) be a topological space. Then (X, T) is $T_0 \Leftrightarrow$ (X, w(T)) is fuzzy T_0.

4. FUZZY R_0-TOPOLOGICAL SPACES

Fuzzy R_0-spaces have been defined by Hutton and Reilly [1] and R. Srivastava, Lal, and A.K. Srivastava [5] as follows :

DEFINITION 4.1(Hutton [1]) An fts (X, τ) is said to be fuzzy R_0 iff each fuzzy open set can be written as a supremum of fuzzy closed sets.

DEFINITION 4.2. (Srivatava [5]). An fts (X,τ) is fuzzy R_0 iff $\forall\, x, y \in X$, $x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(y) = 1$ and $V(x) = 0$.

It has been shown in Srivastava [5] that Definition 4.1 and Definition 4.2 are totally independent and that the latter definition is a good extension of the concept of an R_0 topological space while the former is not. We propose here another definition of fuzzy R_0-spaces depending upon fuzzy points rather than ordinary set points as given in Definition 4.2.

DEFINITION 4.3. An fts (X,τ) is fuzzy R_0 iff for all distinct fuzzy points p, q in X whenever there is a $U \in \tau$ such that $p \in U$ and $q \notin U$, there is also $V \in \tau$ such that $q \in V$ and $p \notin V$.

THEOREM 4.1. For an fts (X,τ) consider the following statements :

(I) For all distinct fuzzy points p, q in X whenever there is a $U \in \tau$ such that $p \in U$ and $q \notin U$, there is also $V \in \tau$ such that $q \in V$ and $p \notin V$.

(II) $\forall$ x, y $\in$ X, $x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(y) = 1$ and $V(x) = 0$. Then statements (I) and (II) are equivalent.

Proof. (I) $\Rightarrow$ (II). Let $x,y \in X$, $x \neq y$ and suppose that there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$. Let p_n, q_n be fuzzy points in X with supports x and y, respectively, and $p_n(x) = q_n(y) = 1 - \frac{1}{n}$, $n \in N$. It is clear that $p_n \in U$ and $q_n \notin U$ for all $n \in N$. Then by (I) $\exists V_n \in \tau$ such that $q_n \in V_n$ and $p_n \notin V_n$, $n \in N$. Let $V = \bigcup_n V_n$, then $V(x) = 1$ and $V(y) = 0$. So we have (II). Conversely, (II) $\Rightarrow$ (I). Let p, q be two distinct fuzzy points in X , $p \neq q$, $p(x) = r$, $q(y) = s$, $r, s \in (0,1)$, and suppose that there is a $U \in \tau$ such that $p \in U$ and $q \notin U$. It is clear that $x \neq y$. Assume that there is $U' \in \tau$ such that $U'(x) = 1$ and $U'(y) = 0$. Then by (II) $\exists V' \in \tau$ such that $V'(y) = 1$ and $V'(x) = 0$. Since $q(y) = s < 1$ then $q \in V'$ and since $p(x) = r > 0$ then $p \notin V'$. Hence, (I) is now implied.

REMARK 4.1. If the fuzzy open sets U and V in Definition 4.3 are replaced by fuzzy closed sets U' and V', respectively, then the statement of theorem 4.1 is still valid.

Following similar arguments as in the proof of Theorem 3.2 we can easily prove the following theorems.

THEOREM 4.2

Let $\{(X_i,\tau_i) : i \in I\}$ be a family of fuzzy R_0-spaces, then the product $(X,\tau) = \prod_i (X_i,\tau_i)$ is fuzzy R_0 iff each coordinates fts is fuzzy R_0.

THEOREM 4.3. A fuzzy subspace of a fuzzy R_0-space is also fuzzy R_0.

THEOREM 4.4. A topological space (X,T) is R_0 iff the fts $(X, w(T))$ is fuzzy R_0.

5. CONCLUSION

It appears more appropriate to define fuzzy T_0 and fuzzy R_0 spaces in terms of fuzzy points rather than ordinary points of a set. It has been shown that the new definition of a fuzzy T_0-space implies the previous ones introduced in [1], [3], and [5]. Also, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary. Moreover, it has been also shown that with these new definitions a topologically generated fuzzy topological space is T_0 or R_0 if the original topological space is T_0 or R_0, respectively. Finally, the main purpose of this note has been grown out of a desire to get definitions of fuzzy T_0 and R_0 spaces in a way that can be extended in a straightforward manner to the case of fuzzy T_1-spaces introduced by Srivastava in [6].

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