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Design and control of Hubbard model in quantum group

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Abstract: Quantum group symmetry for Hubbard model in one dimension is shown to be exist. The Hubbard Hamiltonian function has some useful symmetries. The symmetry which is used in this article restricts to spatial symmetry and dealing with spin. Also, the action of the direct product of magnetic and superconductive of two separates unitary groups $SU(2) \times SU(2)$ is discussed. Recent work serves some of elementary physics and mathematics properties of Hubbard model and its symmetry in quantum group which is an important task.

keywords: Hubbard model; Symmetry; Quantum group; Strong interactions; electrons

1.Introduction

Hubbard J. [7], at the beginning of 1960 proposed a model to describe interactions between electrons (e) in transition–metal monoxides and its correlation in narrow energy bands. This model is strongly interactions of (e) system. Specially, in physics it is open the field to discuss insulating, magnetic, and even novel superconducting effects in a solid by (e) under high temperature. A solid consists of (e) and ions (n) in three dimensions. All particles in physics of a solid at steady stats described by Schrödinger's equation

$$H\,\Psi = E\,\Psi = \;(K+U)\,\Psi \qquad (1)$$

Where, $(H, \Psi,\ E,\ K \ and\ U)$ are respectively, the Hamiltonian function, the wave function, the energy, Kinetic energy, and potential energy. In a solid where (e) can move around, the interaction between electrons as a screened Coulomb interaction. The biggest interaction will be for two electrons on the same site.

In section 2, the creation and destruction operators in the Hubbard model is shown which differ from the operators of harmonic oscillator. In section 3, the symmetry of HH is discussed in one dimension using the product of magnetic and superconductive of two separates $SU(2)$ as $SO(4) = \big(SU(2) \times SU(2)\big)/Z_2$ ([1], [3], [5] and [7]).

2. The Hamiltonian of Hubbard model (HH)

The harmonic oscillator has two operators: the first is creation $\hat{r}^\dagger$ and the second is annihilation $\hat{r}$, which parallel those of the operators in HH. The two operators $\hat{r}^\dagger$ and $\hat{r}$ are defined in terms of the position $(\hat{x})$ and momentum $(\hat{P})$ as

$$\hat{r} = \sqrt{\frac{mw}{2\hbar}}\;\hat{x} + i\sqrt{\frac{1}{2mw\hbar}}\;\hat{P} \qquad (2.a)$$

$$\hat{r}^\dagger = \sqrt{\frac{mw}{2\hbar}}\;\hat{x} - i\sqrt{\frac{1}{2mw\hbar}}\;\hat{P} \qquad (2.b)$$

From $[\hat{P}, \hat{x}] = -i\hbar$, equation (2) obeys the commutation relations

$$[r, r^\dagger\,] = 1 \qquad (3)$$

$$\hat{H} = \frac{1}{2m}\;\hat{P} + \frac{1}{2}\;mw^2\,\hat{x}^2$$
$$= \hbar w\left(\hat{r}^\dagger\,\hat{r} + \tfrac{1}{2}\right) \qquad (4)$$

The Hamiltonian function of these operators can be written as $\hat{H} = \hbar w\left(\hat{n} + \frac{1}{2}\right)$ where $\hat{n} = \hat{r}^\dagger\hat{r}$ is the number operator. Quantum oscillator has ground (vacuum or empty) state $|0>$ and excited stats which are built up by applying $\hat{r}^\dagger$ repeatedly to the ground state $|n>$ as

$$\hat{r}\,|0> = 0 \qquad and$$

$$\hat{H}\,|0> = \frac{\hbar w}{2}\,|0> \qquad (5)$$

$$\hat{r}^{\dagger}\,|n> \;=\; \sqrt{n+1}\;\,|n+1> \quad (6)$$

$$\hat{H}\,|n> = \hbar w\left(n+\tfrac{1}{2}\right)\,|n> \quad (7)$$

he creation and annihilation (destruction) ' fermion ' operators in HH differ from $\hat{r}^{\dagger}, \hat{r}$ for a single harmonic oscillator. In addition to the fact fermion operators are not written in terms of $\hat{x}$ and $\hat{P}$ (position and momentum) but by attaching a site index i and spin index $\sigma \in \{\uparrow,\downarrow\}$ defined as $\hat{a}^{\dagger}{}_{i\sigma}$ $(\hat{a}_{i\sigma})$ create (annihilate) fermions operators. These operators characterized by $n_{i\sigma}$ written as $|\, n_{1\uparrow}\; n_{2\uparrow}\; n_{3\uparrow}\; n_{1\downarrow}\; n_{2\downarrow}\; n_{3\downarrow}\;\; \cdots >$. They are defined to have certain anti-commutation relations (i.e., $\{\hat{X},\hat{Y}\} = \hat{X}\,\hat{Y} + \hat{Y}\,\hat{X}$)

$$\{\hat{a}_{i\sigma_1}\,,\, \hat{a}^{\dagger}{}_{i\sigma_2}\} = \delta_{i,1}\,\delta_{\sigma_1\sigma_2} \qquad (8.a)$$

$$\{\hat{a}^{\dagger}{}_{i\sigma}\,,\, \hat{a}^{\dagger}{}_{1\sigma_2}\} = \{\hat{a}_{i\sigma}\,,\hat{a}_{1\sigma_2}\} = 0 \quad (8.b)$$

Where $\hat{a}^{\dagger}{}_{i\sigma}\,|0> = |1>$ creates a fermion when acting on the vacuum, $\hat{a}^{\dagger}{}_{i\sigma}|1 \geq \hat{a}^{\dagger}{}_{i\sigma}\,\hat{a}^{\dagger}{}_{i\sigma}\,|0> = 0$ which is called Pauli principle since $\hat{a}^{\dagger}{}_{i\sigma}\,\hat{a}^{\dagger}{}_{1\sigma} = -\,\hat{a}^{\dagger}{}_{1\sigma}\,\hat{a}^{\dagger}{}_{i\sigma}$.

The motion and interactions of electrons in a solid describe the HH. We use ' up ' and ' down' to refer to the two fermionic types. The interactions for two electrons on the same site are very big. The HH stops just there: if the site is empty, the interactions are modeled by Zero of fermions or has a single fermion; but has U if the site is occupied by $U_{n_{i\uparrow}n_{i\downarrow}}$.

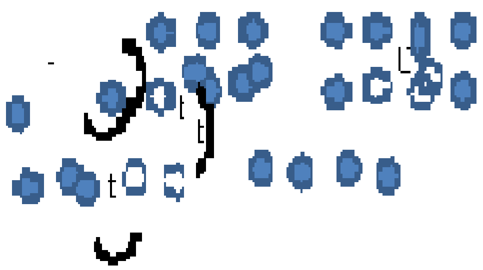

Fig (1): Left (The Kinetic energy K)

Right (The interaction energy I)

In the simplest HH, the interaction between fermions on different sites 1 and i is zero. There exists a kinetic energy on one site and creates it on a neighbor which the HH is illustrated in Fig 1 and defined as

$$\hat{H} = \hat{H}^{non-local}_{(e)} + \hat{H}^{local}_{(e)}$$

$$= K + (I + C) \qquad (9)$$

Where $K = \hat{H}^{non-local}_{(e)}$ is the kinetic energy, I is the interaction (strength) energy and C is a chemical potential which controls the filling. Then, we can write HH dropping the 'hats' of operators as $a\,, a^{\dagger}$

$$\hat{H} = -t\sum_{<i,1>\sigma}(a^{\dagger}{}_{i\sigma}a_{1\sigma} + a^{\dagger}{}_{1\sigma}a_{i\sigma}) + U\,\sum_i n_{i\uparrow}n_{i\downarrow} - \mu\sum_i(n_{i\uparrow} + n_{i\downarrow}) \qquad (10)$$

3 Design of Hubbard model's symmetry in quantum group

The HH has some of useful symmetries. In this article we use symmetry which restrict to spatial symmetry and dealing with spin. The permutation operator at two sites i and j is defined as in [5]

$$p_{i\,,j} = 1 - \left(a^{\dagger}{}_i - a^{\dagger}{}_j\right)\left(a_i - a_j\right)$$

From canonical commutation relations we can write

$$p_{i\,,j}\; a_i = a_j\; p_{i\,,j}$$

The $p_{i\,,j}$ form a representation of symmetric group [6]. For l sites, the operators can be generalized to be spin dependent as

$$p_l = p_{n\uparrow,n-1\uparrow}\, p_{n-1\uparrow,n-2\uparrow} \cdots\; p_{3\uparrow2\uparrow}\, p_{2\uparrow1\uparrow}$$

Where $p_{i\sigma_1\,,j\sigma_2}$ permutes a fermion with spin σ_1 at site i and a particle with spin σ_2 at site j. In this section, we use the standard HH (in one dimension) which has much symmetry like translation or under spin flips as

$$\hat{H} = -t\sum_{<i,j>\sigma} a^{\dagger}{}_{i\sigma}a_{j\sigma} + U\,\sum_i n_{i\uparrow}n_{i\downarrow} - \mu\sum_{i,\sigma} n_{i\sigma} \qquad (11)$$

Where $<i,j>$ the summation over the nearest neighbors. From equation (9) we can write $\hat{H}$ in one dimension as

$$\hat{H} = \hat{H}_{\uparrow} + \hat{H}_{\downarrow} + I \qquad (12)$$

$$\hat{H}_{\uparrow} = -t\sum_{<i,j>\sigma} a^{\dagger}{}_{i\uparrow}a_{j\uparrow} + U\,\sum_i n_{i\uparrow} < n_{i\downarrow} > \qquad (13)$$

$$\hat{H}_{\downarrow} = -t\sum_{<i,j>\sigma} a^{\dagger}{}_{i\downarrow}a_{j\downarrow} + U\,\sum_i < n_{i\uparrow} > n_{i\downarrow} \quad (14)$$

$$I = U\sum_i < n_{i\uparrow} > < n_{i\downarrow} > \quad (15)$$

Hence, the Hamiltonian function is reduced to two matrices $N \times N$ (local densities of system size). In present work, we use size 2×2 to determine the ground state. Several authors [2,4] have shown that HH has $SO(4)$ symmetry at half-filling. This type of symmetry can be obtained from an ordinary symmetry $a_\uparrow^\dagger \leftrightarrow a_\downarrow$ via a "twist" which is not an equivalence transformation but under spin flips.

Frist, the representation of the Lie algebra $su(2)$ which generates the group $SU(2)$ of rotations in spin space with index $\sigma \in \{\uparrow,\downarrow\}$ has a basis (2×2) matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (16)$$

Where $\sigma_i = \{\sigma_x,\ \sigma_y,\ \sigma_z\}$ is called the Pauli matrices satisfying the relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k\ ,\ \ i,j = x,y,z \qquad (17)$$

Where ε_{ijk} is antisymmetric tensor.

Since $SO(4) \cong SU(2) \times SU(2)/Z_2$, the generators of $SO(4)$ symmetry can be written as

$$Y_m^+ = a^\dagger{}_\uparrow a_\downarrow\ ,$$

$$Y_m^- = (Y_m^+)^\dagger = a^\dagger{}_\downarrow a_\uparrow\ , \qquad (18)$$
$$H_m = a^\dagger{}_\uparrow a_\uparrow - a^\dagger{}_\downarrow a_\downarrow$$

Where m is a magnetic of $SU(2)$, $n_\uparrow = a^\dagger{}_\uparrow a_\uparrow$ $n_\downarrow = a^\dagger{}_\downarrow a_\downarrow$. Also, the superconductive of $SU(2)$ is

$$Y_s^+ = a^\dagger{}_\uparrow\, a^\dagger{}_\downarrow\ ,$$
$$Y_s^- = (Y_s^+)^\dagger = a_\downarrow a_\uparrow\ , \qquad (19)$$
$$H_s = a^\dagger{}_\uparrow a_\uparrow + a^\dagger{}_\downarrow a_\downarrow - 1$$

Then

$$H_m = n_\uparrow - n_\downarrow\ , H_s = n_\uparrow - n_\downarrow - 1$$

The symmetry of Z_2 which comes from conjugation does not change the algebra of $a^\dagger{}_i \in \{a^\dagger{}_\uparrow\ ,\ a^\dagger{}_\downarrow\}$ and $a_i \in \{a_\uparrow\ , a_\downarrow\}$ which are fermionic. In equation (18) and (19), the generators give another unitary representation of s for $SU_q(2)$ as

$$[Y_s^+, Y_s^-] = \frac{\left(q_n^{H_s} - q_n^{-H_s}\right)}{q - q^{-1}} \qquad q \in \Re\backslash\{0\},$$
$$[H_s\ , H_s{}^\pm] = \pm\, 2\, H_s{}^\pm \qquad (20)$$

Where $SU_q(2)$ is a deformation of $SU(2)$.

For a single lattice site, equations (18) and (19) hold.

For $q \neq 1 \rightarrow ((H_m)^3 = H,\ (H_s)^3 = H)$

Equation (9) of HH function can be modified to include phonons as [3]:

$$H_{Hub}^* = u \sum_i n_{i\uparrow} n_{j\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma}$$
$$-\lambda \cdot \sum_{i,\sigma} n_{i\sigma}\, x_i + \sum_i \left[\frac{\left(\hat{P}_i\right)^2}{2m} + \frac{1}{2} m w^2 \hat{x}_i{}^2\right]$$
$$+ \sum_{(i,j),\sigma} T_{i,j}\ a^\dagger{}_{ij,\sigma} a_{i\sigma} + h.c.$$

Where $T_{i,j} = t\, e^{[ik\cdot(\hat{P}_i - \hat{P}_j)]}$ (i.e., H_{Hub}^* is invariant under $SU_q(2) \times SU_q(2)/Z_2$ which provided that $\mu = \frac{u}{2} - \lambda^2/mw^2$, $\lambda = hmw^2k$ which will be shown in this section).

Now, the generators of superconductive in one dimension of $SU_q(2)$ will be

$$Y_s^+ = S^+ = a^\dagger{}_\uparrow\, a^\dagger{}_\downarrow = e^{-i\phi P}\ b^\dagger{}_\uparrow\, b^\dagger{}_\downarrow\ ,$$

$$Y_s^- = S^- = a_\downarrow a_\uparrow = e^{i\phi P}\, b_\downarrow b_\uparrow\ , \qquad (21)$$
$$H_s = 2S^{(z)} = n_\uparrow - n_\downarrow - 1$$

and

$$b_{i\sigma} \equiv U(k) a_{i\sigma} U^{-1}(k)$$

$$b^\dagger{}_{j\sigma} \equiv U(k) a_{j\sigma} U^{-1}(k)$$

with unitary operator $U(\alpha) = e^{(i\alpha \cdot \sum_{1,\sigma} P_1 n_{1\sigma})}$ where $b_{i\sigma}$, $b^\dagger{}_{j\sigma}$ depends on a set of constant parameters α_k, $k = 1, \cdots$. Hence, this transformation does not change the value of $n_\uparrow$

and $n_\downarrow$, then the results in an exponential factor in $\hat{P}_i - \hat{P}_j$ for

$$a^\dagger{}_{j\sigma} a_{i\sigma} = e^{ik\cdot(\hat{P}_i - \hat{P}_j)}\, a^\dagger{}_{j\sigma} a_{i\sigma}$$

The symmetry of HH requires $\phi = 2k$ and $\alpha = k$. The co-multiplication in $SU_q(2)$ is

$$\Delta_q(S^\pm) = S^\pm \otimes q^{-H/2} + q^{H/2} \otimes S^\pm$$

$$\Delta_q(S^-) = \left(\Delta_q(S^+)\right)^\dagger$$

$$\Delta_q(H) = H \otimes 1 + 1 \otimes H$$

In present work, we assume $q = e^\alpha$ be a deformation parameter and α is a complex parameter which is determined by the commutation relations. Then, the symmetry of H_{Hub}^* requires [1], [8] and [9]

$$\phi = \frac{2\lambda}{hmw^2}$$

$$\mu = \frac{u}{2} - \frac{1}{4}mw^2h^2\phi^2 = \frac{u}{2} - \frac{\lambda^2}{mw^2}$$

To obtain

$$\left[S^+, H^{(loc)}\right] = \left[S^-, H^{(loc)}\right] = \left[S^{(z)}, H^{(loc)}\right] = 0$$

3. Results and Discussion

There are many more studies of HH in Physics than in mathematics. In mathematics, we are interested in the study of quantum group symmetry. quantum group symmetry of HH has interesting features like phonons which naturally can be included without breaking the concept of symmetry under high temperature. Also, symmetries of magnetic and superconducting are included. There exists a difference between H and H^*_{Hub} that H^*_{Hub} contains coordinates with $(b_{i\sigma}, {b^\dagger}_{j\sigma})$ while H contains coordinates with $({a^\dagger}_{j\sigma}, a_{i\sigma})$ is shown in detail. A full quantum symmetry is shown with all elements of $SU_q(2)$.

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