Solution of Parapolic Navier-Stokes Equations for Laminar Forced Convective Flow in Entrance Region of a Flat Passage

حل معادلات نافير-ستوكس المكافئة للحمل الجبري الرقائقي في منطقة الدخول لممر مستو

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خلاصة: نظرا لأهمية سريان المائع على عملية انتقال الحرارة بالحمل الجبري فانه فى هذا البحث تم وصف سريان المائع بمعادلات نافير-ستوكس المكافئة وتم حلها بواسطة طريقة الحل الغير تشابهي. طبقا لهذه الطريقة وبالتعريف المناسب للمتغيرات فان المعادلات الواصفة للمسألة (معادلات الحركة ومعادلة الطاقة) تتحول الى مجموعة من المعادلات التفاضلية العادية. يتم حل هذه المجموعة من المعادلات بطريقة رانج-كوتا مصحوبة بطريقة الرصد لمسائل القيمة الحدية.
وهكذا امكن حساب رقم نوسلت المحلى وكذلك معامل الاحتكاك المحلى للموائع ذات رقم براندتل المساوى للواحد الصحيح. تم دراسة مجموعة من الممرات ذات ارتفاعات مختلفة تبعا لقيمة رقم رينولدز(١٠٠، ٢٠٠، ٣٠٠). فى هذه الدراسة تم اقتراح علاقتين رياضيتين لرقم نوسلت للممرات المختلفة.

Abstract- Due to dependence of forced convective heat transfer on the hydrodynamic flow field; any improvement on the analysis of this flow field is of great importance to understand the heat transfer process. In this work the flow field is described by the parabolic Navier-Stokes equations; which are solved by local non-similarity solution-method. According to this method; with suitable definition of the problem variables , a set of ordinary differential equations are produced. This set is solved, numerically, by Runge-Kutta method accompanied with shooting method of boundary value problems.

The values of local Nussell number and local coefficient of friction are calculated for fluids of Prandtl number of one. Some passages of Reynolds number(Re_L) of 100, 200 & 300 are studied here. Two formulae of Nussell number for different passage heights are proposed.

1. Introduction

Good understanding of convective heat transfer problems is dependent, principally, on good analysis of the hydrodynamic flow field. The developement of hydrodynamic flow field in combined entrance region of a duct was studied by many investigators. Kakac and Yener [2] surveyed different methods developed to solve the problem of laminar forced convection in combined entrance region of a duct. Wasel [4] made a local similarity solution of laminar forced convection in entrance region for flow between two parallel plates.

In those works, the hydrodynamic flow field was described by boundary layer equations. Recently, Wasel [5] made a solution of the flow field based on the parabolic Navier-Stokes equations [1]. According to the used method in his solution some terms in modified governing equations were dropped. In present work, an improvement of his solution is made by solving the governing equations of the problem by local non-similarity solution method [3]. Because of the nature of the governing equations and of the used technique, the solution is carried out in step by step manner.

2. Governing Equations

As shown in Fig. (1) the laminar flow between two parallel flat plates is considered. The uniform velocity of approach, the temperature and pressure at inlet of the passage are denoted as u_o, T_o & P_o, respectively. The height of the passage is taken as 2b. Wall temperature (T_w) is assumed to be constant. Constant fluid properties are assumed.

The governing equations can be written in Cartesian co-ordinate x, y as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad , \qquad (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2} \quad , \qquad (3)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \quad , \qquad (4)$$

where u and v are the velocity components in x- and y-directions, respectively, T and p are the temperature and pressure of fluid. ρ, ν and α are the density, kinematic viscosity and thermal diffusivity, respectively. The second derivative of u,v and T with respect to x are assumed to be small compared with other terms in equations (2)-(4).

The velocity profile at any position (x) must satisfy the continuity equation in integral form and hence, one can write the equation;

$$\int_0^b u\, dy = u_o\, b \quad . \qquad (5)$$

According to equation (5), the velocity at axis of similarity must be corrected to the proper value, which produces a velocity profile satisfies this equation.

Due to the similarity of hydrodynamic as well; thermal fields about the axis of the passage, it is convenient to solve the governing equations from one wall to the center of the passage. Equations (1)-(4) possess the following boundary conditions;

$$u = v = 0 \; ; \; T = T_w \quad \text{at } y = 0 \quad , \tag{6-a}$$

$$u = u_{o,x} \; ; \; \frac{\partial u}{\partial y} = 0 \; ; \; T = T_o \quad \text{at } y = b . \tag{6-b}$$

To express the governing equations in dimensionless form, one introduces new independent variables ξ , η as follows;

$$\xi = \frac{1}{b}\sqrt{\frac{x\,\nu}{u_o}} \quad , \; \eta = y\sqrt{\frac{u_o}{\nu\,x}} \quad . \tag{7}$$

Furthermore, a dimensionless forms of stream function, pressure and temperature are defined according to the following relations;

$$\psi(x,y) = \sqrt{u_o\,x\,\nu}\; f(\xi,\eta) \quad , \tag{8-a}$$

$$P(\xi,\eta) = (\,p(x,y) - p_\infty\,) \,/\, \rho\, u_\infty^2 \quad , \tag{8-b}$$

$$\theta(\xi,\eta) = (\,T - T_\infty\,) \,/\, (\,T_w - T_\infty\,) \quad , \tag{8-c}$$

where $\psi(x,y)$ is the stream function, which is defined such that it satisfies the continuity equation (1). $f(\xi,\eta)$, $P(\xi,\eta)$ and $\theta(\xi,\eta)$ are the dimensionless forms of stream function, pressure and temperature, respectively. Substitution of equations (7)-(8) into equations (2)-(4) leads to the following dimensionless form of governing equations,(where the primes denoting differentiation with respect to η and the suffix ξ denoting the differentiation with respect to ξ) :

$$2\,f''' + f\,f'' + \eta\,P' = \xi\,f'_\xi\,f' - \xi\,f_\xi\,f'' + \xi^2\,P_\xi \quad , \tag{9}$$

$$\begin{aligned} 2\,\eta\,f''' + 2\,f'' + \eta\,f\,f'' + f\,f' - \eta\,f'^2 - 4\,\xi^2\,Re_b^2\,P' = {} & \xi\,f'\,f_\xi + \xi\,f\,f'_\xi + \xi^2\,f_\xi\,f'_\xi - \xi\,f_\xi\,f'' - \\ & \eta\,\xi\,f'\,f'_\xi + 2\,\xi\,f''_\xi \quad , \end{aligned} \tag{10}$$

$$\frac{2}{Pr}\,\theta'' + (\,f + \xi\,f_\xi\,)\,\theta' - \xi\,f'\,\theta_\xi = 0 \quad , \tag{11}$$

where Re_b is the Reynolds number based on the half of the passage height and defined as $Re_b = u_o\,b\,/\,\nu$.

Defining G as the derivative of f with respect to ξ and neglecting P_ξ from equation(9), one can eleminate P from equations (9)-(10). The produced equation is written in simple appearance as follows ;

$$a_1 f''' + a_2 f'' + a_3 f' + a_4 f + a_5 = 0 \quad , \quad (12\text{-a})$$

where a_1, a_2, a_3, a_4 and a_5 are coefficients intorduced to put the obtained equation in simpler appearance. These coefficients have the following definitions ;

$$a_1 = 8\,\xi^2 Re_b^2 + 2\,\eta^2 \quad , \quad (12\text{-b})$$

$$a_2 = 4\,\xi^2 Re_b^2 f + \eta^2 f + 2\,\eta + 4\,\xi^3 Re_b^2 G + \xi\,\eta\,G \quad , \quad (12\text{-c})$$

$$a_3 = -(\,4\,\xi^3 Re_b^2 G' - \eta f + \eta^2 f' + \xi\,\eta\,G - \eta^2 \xi\,G'\,) \quad , \quad (12\text{-d})$$

$$a_4 = -\,\xi\,\eta\,G' \quad , \quad (12\text{-e})$$

$$a_5 = -\,(\,\xi^2 \eta\,G\,G' + 2\,\xi\,\eta\,G''\,) \quad . \quad (12\text{-f})$$

Accordind to the used technique of solution (local non similar solution-method [3]), the derivative of f with respect to ξ is dealed with as a new dependent variable (G) and thus one needs another subsidiary equation. This equation can be obtained by defferentiating equation (12) with respect to ξ. The produced equation takes the following form;

$$b_1 G''' + b_2 G'' + b_3 G' + b_4 G + b_5 = 0 \quad , \quad (13\text{-a})$$

where the coefficient b_1, b_2, b_3, b_4 and b_5 are defined as ;

$$b_1 = 8\,\xi^2 Re_b^2 + 2\,\eta^2 \quad , \quad (13\text{-b})$$

$$b_2 = 4\,\xi^2 Re_b^2 f + \eta^2 f + 4\,\xi^3 Re_b^2 G + \xi\,\eta\,G \quad , \quad (13\text{-c})$$

$$b_3 = -(\,4\,\xi^3 Re_b^2 G' + \eta^2 f' - \eta^2 \xi\,G' + 4\,\xi\,\eta\,G + 12\,\xi^2 Re_b^2 f'\,) \quad , \quad (13\text{-d})$$

$$b_4 = 16\,\xi^2 Re_b^2 f'' + \eta^2 f'' + \eta f'' \quad , \quad (13\text{-e})$$

$$b_5 = 16\,\xi\,Re_b^2 f''' + 8\,\xi\,Re_b^2 f\,f'' \quad . \quad (13\text{-f})$$

According to the definition of G and by neglecting the derivative of θ with respect to ξ, the energy equation(11) takes the following form ;

$$\frac{2}{Pr}\,\theta'' + (\,f + \xi\,G\,)\,\theta' = 0 \qquad (14)$$

Equations (12)-(14) represent a system of ordinary differential equations in f, G and θ as unkowns and ξ is dealed with as a parameter. This system of equations has the following boundary conditions ;

$$f = f' = G = G' = 0 \quad ; \quad \theta = 1 \quad \text{at} \quad \eta = 0 \quad , \qquad (15\text{-a})$$

$$f'' = G'' = \theta = 0 \quad ; \quad f' = u_{0,x} / u_0 \ \text{at} \ \eta = \eta_b \quad , \qquad (15\text{-b})$$

where f' at η_b has to produce a velocity profile satisfies the following condition ;

$$\int_0^{\eta_b} f'\, d\eta = \eta_b \quad , \qquad (15\text{-c})$$

where η_b is the value of the variable η at the center of the passage ($\eta_b = b\sqrt{u_o / \nu x}$). Equation (15-c) is derived using the definitions of dimensionless variables; equations (7-8) & equation (5).

3. Numerical Procedure

According to the local non-similarity method [3], the modified governing equations (12)-(15) are solved for different values of the parameter ξ. As it is clear from equation (7) the value of ξ can be expressed in terms of η_b ($\xi = 1/\eta_b$), thus the solution of the governing equations is carried out several times for different values of η_b and in turn; for different values of ξ.

In case of forced convective flow, the momentum equations (12)-(13)& (15) can be solved separately and then the solution of energy equation (14) can be carried out. For certain value of ξ, the mumerical solution is carried out through two main steps. First; the considered equations(12-13& 15) are solved for assumed values of $f''(0)$& $G''(0)$ by Runge-Kutta method of ordinary differential equations and then the solution is corrected by shooting method of boundary value problems to satisfy the boundary conditions at $\eta=\eta_b$ ($f''(\eta_b = G''(\eta_b) = 0$). Second step; is to correct the obtained velocity profile (f' versus η) to satisfy the equation (15-c). This is achieved by using shooting method for second time.

Knowing the value of f& G for different values of η; as it is done before, the energy equation and its boundary conditions equations (14)-(15) can be solved in similar manner.

When the fields of velocity and temperature have been obtained, local Nusselt number (Nu_w) and local coefficient of friction (C_f) can be determined according to the following definitions;

$$\tau_w = \rho \nu \frac{\partial u}{\partial y})_{y=0} \quad , \qquad (16)$$

$$q_w = h (T_w - T_o) = - k \frac{\partial T}{\partial y})_{y=0} \quad , \qquad (17)$$

where q_w is the heat flux at single plate of the two heated plates. Introducing the dimensionless variables in equations (16)-(17), one obtains the following expressions of local Nusselt number and local coefficient of friction;

$$Nu_x / \sqrt{Re_x} = \theta'(\xi, 0) \quad , \qquad (18)$$

$$C_f' \sqrt{Re_x} = f''(\xi, 0) \quad , \qquad (19)$$

where Re_x denotes the local Reynolds number ($u_o x / \nu$) and Nu_x is local Nusselt number ($h x/k$).

4. Results and Discussion

The calculations are carried out for three different passage heights according to the value of Re_b ($u_o b/\nu$) equals to 100,200& 300. Through these calculations the fluids of Pr=1.0 is considered.

Fig.(2) shows the relation between local coefficient of friction and local Nusselt number as defined in equations (18)-(19) versus the dimensionless distance along the passage represented by ξ. It is clear that no effect of passage height on the value of coefficient of friction at values of ξ>0.15. For Nusselt number, in the same figure ;in general, the height of passage has no effect or at least very small effect through out the studied range of ξ. Consequently it is convenient to derive a formula for Nusselt number represented by $\theta'(0,\xi)$ as a function of distance along the passage wall represented by ξ. Fig.(3) shows two proposed formulae compared with the calculated data points. They, mathematically, are expressed in the two following relations as;

$$\theta' = 0.302836 \, \xi + 0.338300 \qquad \text{(linear fit)}$$

$$\theta' = 0.340181 \, \text{Exp} \, (0.782064 \, \xi) \qquad \text{(exponential fit)}$$

According to the definition of Nusselt namber as well; coefficient of friction equations (18)-(19) and other variables of the problem, one can obtain new relations after some manipolations of the obtained numerical results (θ'& f'' versus ξ). Fig.(4) shows the relations between Nu_x and the dimensionless distance along the passage [$(x/b)/Re_b$ Pr].

Nusselt number (Nu_x) increases rapidly near the entrance of the passage then it increases linearly for greater distance. Fig.(5) shows local Nusselt number based on the half of passage height (Nu_b= h b/ k) versus dimensionless distance along the plate (x/b). Near the entrance of the passage Nu_b has very large values and suddenly drops and goes to asymptotic values (as it is clear in case of Re_b= 300). Fig.(6) shows local coefficient of friction versus dimensionless distance (x/b) for different values of passage heights. For narrow passage (Re_b= 100) the value of coefficient of friction is greater.

5. Conclusion

the used technique in present work, presents a simple possible way to deal sufficiently with parabolic Navier-Stokes equations. In the same time, this technique is suitable to be used to solve the energy equation. According to the proper transformation of the variables of problem, two proposed formulae for estimating Nusselt number of considered problem are presented here.

6. Nomenclature

2b	the passage height
C_f	coefficient of friction, $\tau_w/\rho u_o^2$
f	dimensionless stream function, $\psi / \sqrt{u_o \nu x}$
h	local heat transfer coefficient, defined by eq.(17)
k	thermal conductivity of fluid
Nu_b	local Nusselt number based on b, hb/k
Nu_x	local Nusselt number, hx/k
Pr	Prandtl number, ν/α
q_w	heat flux at the wall
Re_b	Reynolds number based on the half of the passage height (b), $u_o b/\nu$
Re_x	local Reynolds number, $u_o x/\nu$
T	temperature of fluid at general position x, y
T_o	temperature of fluid at the inlet cross-section
T_w	wall-temperature
u	velocity component in x direction
u_o	velocity at the inlet of the passage
$u_{o,x}$	the velocity at the center of the passage at any value of x
v	velocity component in y-direction

x	co-ordinate along the lower wall
y	co-ordinate normal to the lower wall of the passage
α	thermal diffusivity, $k/\rho c_p$
η	dimensionless independent variable, $y\sqrt{u_o/x\nu}$
η_b	the value of η at the center of the passage
ξ	dimensionless independent variable, $\frac{1}{b}\sqrt{x\nu/u_o}$
ν	fluid kinematic viscosity
ρ	fluid density
θ	dimensionless form of temperature, $(T-T_o)/(T_w-T_o)$
τ_w	wall shear stress in x-direction
ψ	stream function

7. Refrences

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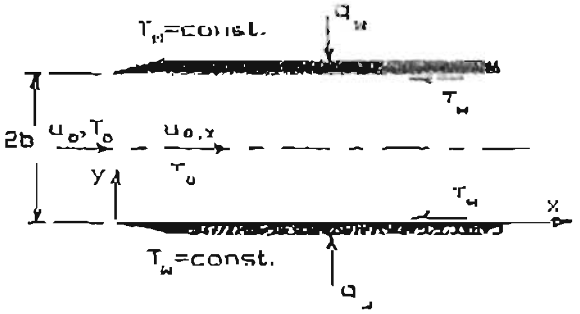

Fig. (1) Schematic description of the flow through the passage.

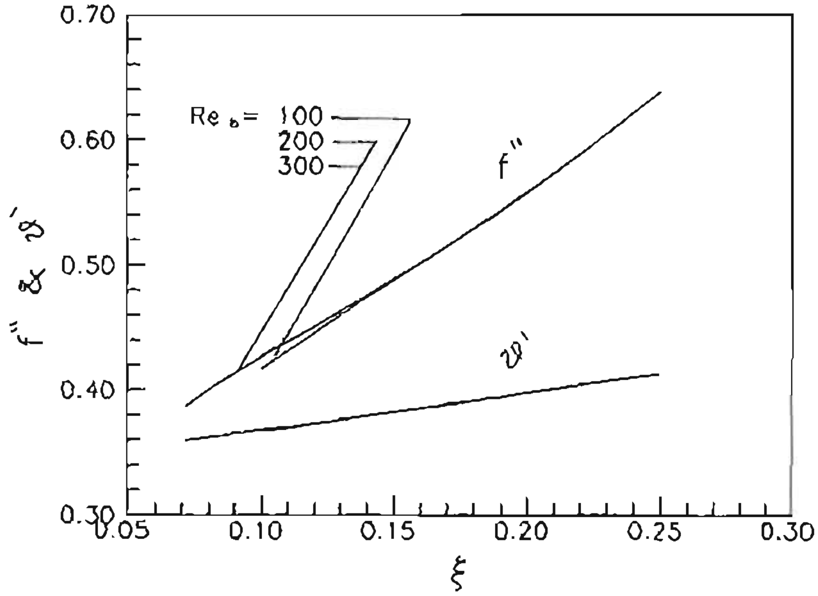

Fig. (2) Nusselt number and coefficient of friction as defined in equations (18& 19) versus the dimensionless distance ξ for Re_b equals to 100,200 & 300.

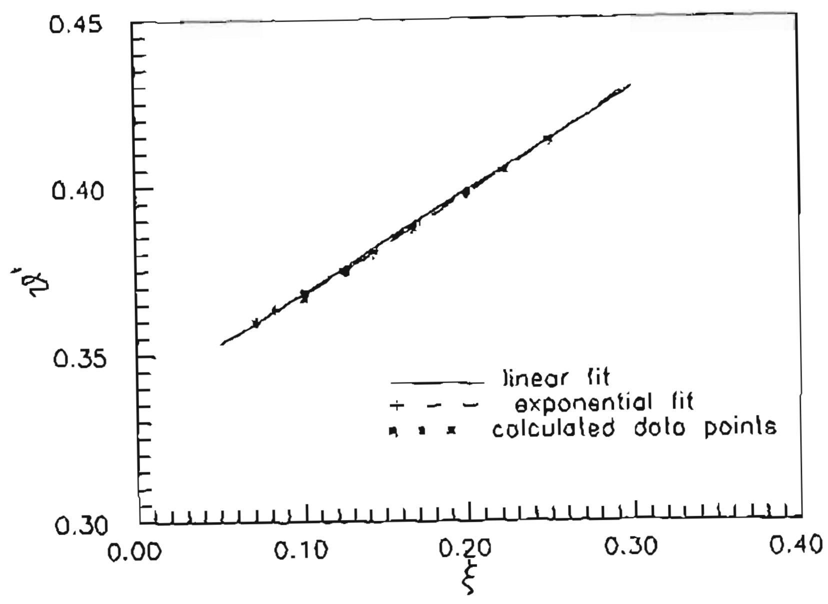

Fig.(3) The calculated Nusselt number as defined in equation (18) compared with linear and exponential regression.

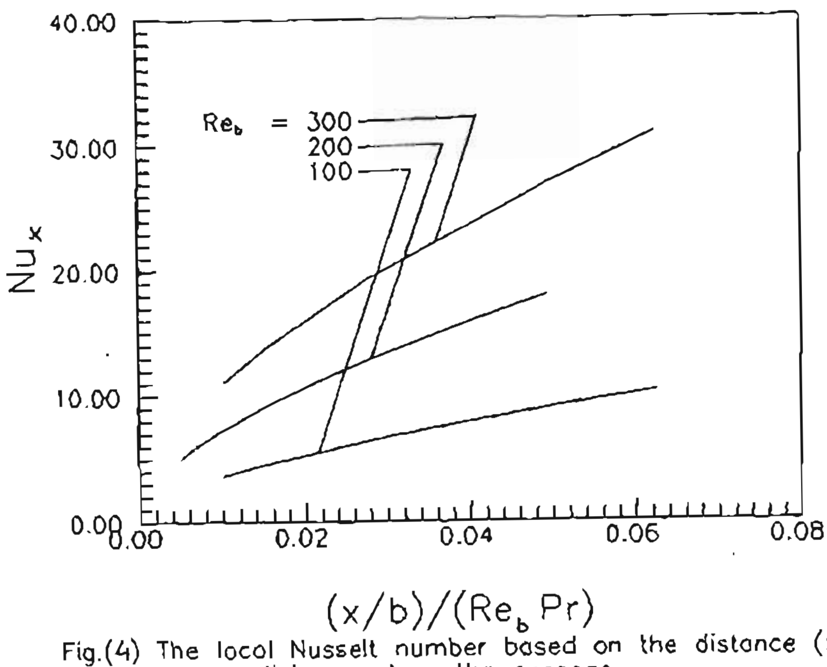

Fig.(4) The local Nusselt number based on the distance (x) versus distance along the passage.

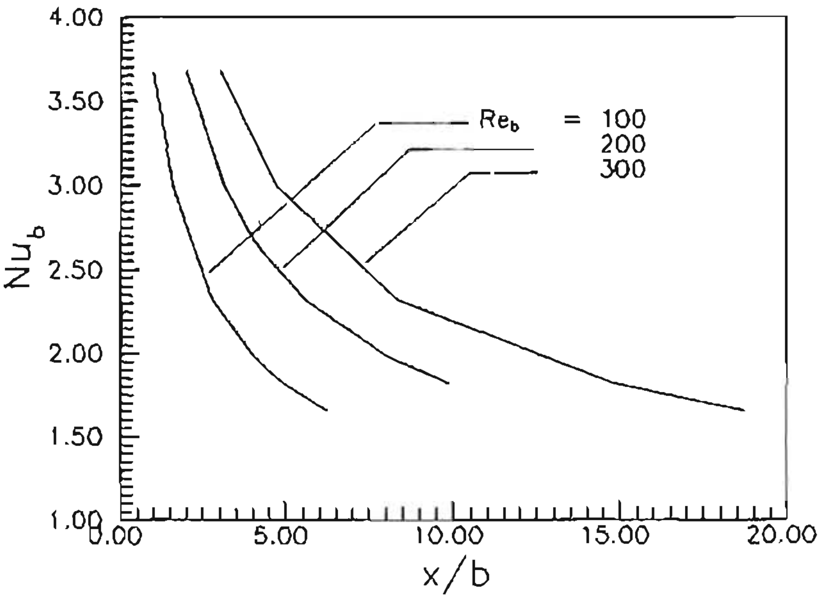

Fig.(5) Nusselt number based on the half of the passage height versus distance along the passage.

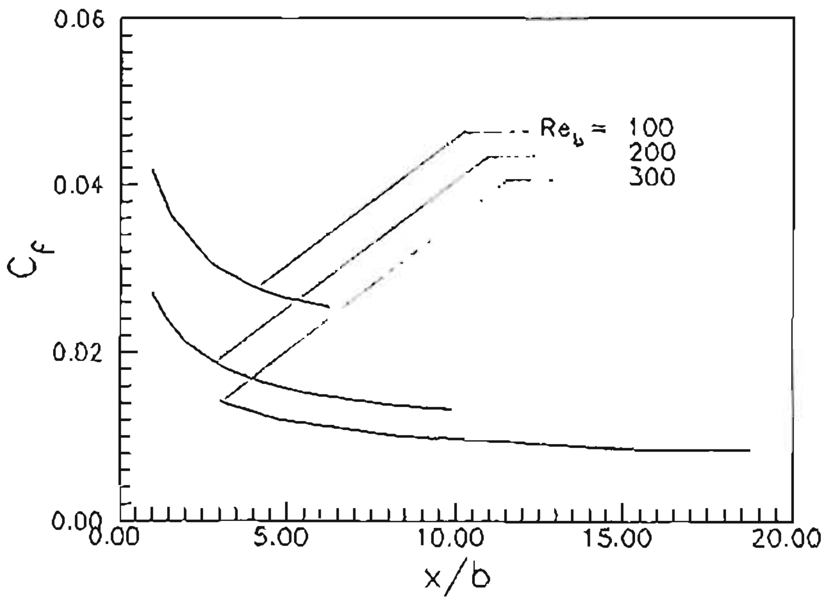

Fig.(6) The coefficient of friction along the passage for different passage height.