Sci. J. Fac. Sci. Monoufia Univ., Vol. II, 1988.

ON THE THEORY OF PLANE COUETTE FLOW WITH POROSITY.

A.M. ABOURABIA

Department of Math., Faculty of Science, Menoufia university Shebin El-Koom, Egypt.

ABSTRACT.

Plane steady Couette flow at low Mach number is studied in the presence of porosity. An approximate solution to the Boltzmann equation, of modified Liu-lees type, is found to yield simple analytic expressions for flow velocity distribution, mean velocity and shear stress. These predictions give good results in both the continuum and rarefied limits.

1) Introduction:

The theory of interaction between gases and solid surfaces is far from being in its final stage. Many authors such as (Shedlovskii 1967), (Kogan 1969), (Chapman & Cowling 1970), (Khidr 1970), (Cercignani 1975), (Hady 1976), (Johnson 1982), (Mahmoud 1985) made successful demonstrations of the gas flow using Boltzmann kinetic equation, specially in the study of Couette compressible flow between two solid parallel walls. Various models had been suggested, but the simplest of them takes the gas-surface interaction in the form of a tangential accommodation coefficient with respect to momentum. Different efforts had been done in imposing the conditions at the boundaries, however they are not enough in describing a variety of phenomena that demand further predictions.

In the works of (Hady 1976) and (Mahmoud 1985) the equations of transfer are used to describe the problem of Couette flow in rarefied gases with porosity, they open a large area of study to follow this effect.

2) Setting up the problem:

In this paper Couette compressible flow in the absence of external force and with porosity is discussed on the assumption that at every point of the flow the gas is in a near-equilibium state. Then the mean velocity and the mean relative velocity of thermal motion are of the same order. But we shall distinguish between them below only for the sake of giving the agreement with the real flows.

For simplicity it will be assumed that the flow is gentle i.e. the Mach number.

$$M = \sqrt{\alpha_e}\, U \ll 1$$

Where $\alpha_o = (2RT)^{-1}$, T is the constant temperature at the walls, $U/2(-U/2)$ is the velocity of the upper (lower) wall in the x-direction, d is the distance between the walls and $y=+d/2(-d/2)$ is the equation of the upper (lower) wall. To investigate the effect of porosity one considers that the gas flows out from the lower and upper walls with velocities $V_1=-aU$ and $V_2=bU; a,b>0$.

The relevant Boltzmann equation governing the present problem is

$$c_y \frac{\partial f}{\partial y} = I(f) \qquad (1)$$

Where I(f) is the usual Boltzmann collision aperator. Because shear is assumed to be week, it is reasonable to linearise the number density distribution function f about the zero-shear Maxwellian:

$$f_\circ = n \ (\alpha_\circ/\pi)^{3/2} \exp[-\alpha_\circ (\underline{c} - \underline{v})^2] \qquad (2)$$

Where n is the constant density number.

To obtain an approximate kinetic-theory solution to eq.(1) one may use the B.G.K. equation with which a molecule tends to relax to local equilibruim after a single collision. Thus

$$c_y \frac{\partial f}{\partial y} = I(f) \rightarrow \frac{\sqrt{2RT}}{l} (f_\circ - f). \qquad (3)$$

i.e. the processes of transfer of molecular quantities depend appreciably on the mean free path l.

One may look for a solution of eq.(3) by the approximation method of (Liu & Lees 1961). The method consists in replacing the exact distribution function by a two stream Maxwellian. For plane Couette flow with no external force and the above mentioned geometry, one has.

$$f = f^- \Theta(-c_y) + f^+ \Theta(+c_y).$$

Where $\Theta(\pm c_y)$ is the Heaviside step function:

$$\Theta(c_y) = \begin{cases} 1 : c_y > 0 \\ 0 : c_y < 0 \\ \tfrac{1}{2} : c_y = 0 \end{cases}$$

and $f^{\mp}$ are chosen to be

$$f^- = n \ (\alpha_\circ/\pi)^{3/2} (1+V_1 \ c_y/RT) \exp -\alpha_\circ[(c_x-v_{x1})^2+c_y{}^2+c_z{}^2] \ : c_y<0.$$
$$f^+ = n \ (\alpha_\circ/\pi)^{3/2} (1+V_2 \ c_y/RT) \exp -\alpha_\circ[(c_x-v_{x2})^2+c_y{}^2+c_z{}^2] \ : c_y>0. \qquad (4)$$

Here v_{x1}, v_{x2} are y-dependent parameters determined by the requirement that f satisfies a suitable number of (lower order) moments of the governing equations.

In this problem, one wants to predict the flow velocity distribution, the mean velocity and the pressure deviator.

3) <u>The boundary conditions:</u>

As it was said in the introduction, we assume diffuse plus specular reflection at the boundaries with coefficients $\varepsilon_1(\varepsilon_2)$ at the lower (upper) wall. Using functions (4), it is desired to integrate four equations of the form.

$$f^{\pm} = (1 - \varepsilon_{1,2})\, f^{\mp} + \varepsilon_{1,2}\, f_{s1,2}$$

with respect to c_x. Where

$$f_{s1,2} = n\,(\alpha_o/\pi)^{3/2}\,[1+V_{1,2}\,c_y/RT]\exp -\alpha_o[(c_x \pm U/2)^2+c_y^2+c_z^2]\ .$$

Hence we obtain, at the lower wall

$$v_{x1}^{-}(-\tfrac{1}{2}) = (1 - \varepsilon_2)\, S\, v_{x2}^{+}(\tfrac{1}{2}) - \tfrac{1}{2}\varepsilon_1. \qquad (5)$$

$$v_{x1}^{+}(-\tfrac{1}{2}) = (1 - \varepsilon_1)\quad v_{x1}^{-}(-\tfrac{1}{2}) - \tfrac{1}{2}\varepsilon_1. \qquad (6)$$

at the upper wall

$$v_{x2}^{+}(\tfrac{1}{2}) = (1-\varepsilon_1)S^{-1}\ v_{x1}^{-}(-\tfrac{1}{2}) + \tfrac{1}{2}\varepsilon_2. \qquad (7)$$

$$v_{x2}^{-}(\tfrac{1}{2}) = (1-\varepsilon_2)\quad v_{x2}^{+}(\tfrac{1}{2}) + \tfrac{1}{2}\varepsilon_2. \qquad (8)$$

Where y and both $v_{x1,2}$, $V_{1,2}$ are nondimensionalized with respect to d and U respectively.

Here $v_x^{\pm}$ indicate the upword and downword velocities in either half space $y \gtrless 0$. The suction factor S is equal to.

$$S = (\tfrac{1}{2} + \gamma b/\pi)(\tfrac{1}{2} + \gamma a/\pi)^{-1}.$$

The solutions of the system of equations (5-8) are obtained in terms of the arbitrary coefficients $\varepsilon_1, \varepsilon_2$ and S;

$$v^-_{x1}(-\tfrac{1}{2})=[\varepsilon_2(1-\varepsilon_2)S-\varepsilon_1][2(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)]^{-1} \quad (9)$$

$$v^+_{x1}(-\tfrac{1}{2})=\{(1-\varepsilon_1)[(1-\varepsilon_2)\varepsilon_2 S-\varepsilon_1\varepsilon_2]-\varepsilon_1\}[2(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)]^{-1} \quad (10)$$

$$v^+_{x2}(\tfrac{1}{2})=[\varepsilon_2 S-(1-\varepsilon_1)\varepsilon_1][2S(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)]^{-1} \quad (11)$$

$$v_{x2}(\tfrac{1}{2})=\{\varepsilon_2 S[1+\varepsilon_1(1-\varepsilon_2)]-\varepsilon_1(1-\varepsilon_1)(1-\varepsilon_2)\}[2S(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)]^{-1} \quad (12)$$

They are supplemented by another set of boundary conditions concerning the mean velocity at the lower and upper walls [Chapman & Cowling 1970]:

$$\langle v_{x1}(-\tfrac{1}{2})\rangle=\tfrac{1}{2}[v^+_{x1}(-\tfrac{1}{2})+v^-_{x1}(-\tfrac{1}{2})]=v^-_{x1}(-\tfrac{1}{2})-u\sqrt{\pi}\,Kn\,\partial v_x/\partial y \quad (13)$$

$$\langle v_{x2}(\tfrac{1}{2})\rangle=\tfrac{1}{2}[v^+_{x2}(\tfrac{1}{2})+v^-_{x2}(\tfrac{1}{2})]=v^+_{x2}(\tfrac{1}{2})+u\sqrt{\pi}\,Kn\,\partial v_x/\partial y \quad (14)$$

Where $\partial v/\partial y$ is the constant velocity rate between the two walls, u is a constant of order unity and the Knudsen number $Kn=\sqrt{\pi}\,l/d$.

4) The characteristics of the flow:

The determination of v_{x1} and v_{x2} are obtained by demanding that two moments of eq.(3) be satisfied, those taken with respect to c_y and $c_x c_y$,

$$v^{\pm}_{x2}-v^{\pm}_{x1}+\gamma(bv^{\pm}_{x2}-av^{\pm}_{x1})=\alpha^{\pm} \quad (15)$$

$$\tfrac{1}{2}(v^{\pm}_{x2}+v^{\pm}_{x1})+\gamma/\pi(bv^{\pm}_{x2}+av^{\pm}_{x1})=\beta^{\pm}-{}^{\pm}\alpha y \quad (16)$$

Where γ is related to the Mach number by the expression $\gamma=\sqrt{\pi}M$ and δ is the degree of rarefaction $=(Kn)^{-1}$.

The constants of integration $\alpha^{\pm}$, $\beta^{\pm}$ should be calculated at the boundaries (the walls) $y=\pm\tfrac{1}{2}$, by solving eqs.(15) and (16) simultaneously using the conditions (9-12), thus

$$v^{\pm}_{x2}(y)=C_1A_1A_3\{\ C_2\ +\ C_3\ [A_2-\delta y\Theta(+y)]\} \tag{17}$$

$$v^{\pm}_{x1}(y)=C_1B_1B_3\{\ C_2\ -\ C_3\ [B_2+\delta y\Theta(-y)]\} \tag{18}$$

Where

$C_1=(A_2+B_2-\delta)^{-1}$, $C^{\pm}_2=(B_2-\delta/2)(A_1A_3)^{-1}v^{\pm}_{x2}(\tfrac{1}{2})+(A_2-\delta/2)(B_1B_3)^{-1}v^{\pm}_{x1}(-\tfrac{1}{2})$,

$C_3=(A_1A_3)^{-1}v_{x2}(\tfrac{1}{2})-(B_1B_3)^{-1}v^{\pm}_{x1}\ (-\tfrac{1}{2})$, $A_1,B_1=[\tfrac{1}{2}+\gamma(a,b)/\pi]$,

$A_2,B_2=(\tfrac{1}{2}+\gamma(a,b)/\pi\ ][\tfrac{1}{2}+\ \gamma(a,b)]^{-1}$ and.

$A_3,B_3=[1+\ \gamma(a,b)][\tfrac{1}{2}+\gamma(a,b)/\pi]\ [1+\ \gamma(b,a)][\tfrac{1}{2}+\gamma(a,b)/_{\pi}]\ +$
$+\ [1+\ \gamma(a,b)][\tfrac{1}{2}+\ \gamma(b,a)/_{\pi}]\ ^{-1}.$

In this study we deal with a problem of nonsymmetrical nature because the differences between ε_1, ε_2 and V_1,V_2 make, for instance, $v^{\pm}_{x2}(y)\neq -v^{\pm}_{x1}(-y)$ this implies that the flow velocity is nonzero at the line y=0.

Of interest are the three following quantities:

(i) <u>The flow velocity distribution function</u>:

The mean relative velocity is defined by

$$\langle v_{12}(y)\rangle\ =\ \langle v_{x2}(y)\ -\ v_{x1}(y)\rangle.$$

It could be correctly written to fullfil our requirements as

$$\langle v_{12}(y)\rangle\ =\ \langle v_{x2}(y)\rangle\ -\ \langle v_{x1}(y)\rangle.$$

This relation in turn is equal to the flow velocity rate $[\partial v_x(y)/\partial y]y$. Therefore, from eqs.(17) and (18) after some manipulations we get.

$$[\partial v_x(y)/\partial y]y=C_1\{\ \langle C_2\rangle\ (A_1A_3-B_1B_3)+\langle C_3\rangle[(A_1A_2A_3+B_1B_2B_3)- -\ \delta\ (A_1A_3\ \Theta(y)-B_1B_3\Theta(-y))y]\}. \tag{19}$$

Integrating eq.(19) with respect to y to obtain the velocity distribution function in plane Couette flow with porosity.

$$v_x(y)=C_1\{D\ \ln y-\delta\langle C_3\rangle[A_1A_3\Theta(y)-B_1B_3\Theta(-y)]y\}+C_4. \qquad (20)$$

Where $D=\langle C_2\rangle(A_1A_3+B_1B_3)+\langle C_3\rangle(A_1A_2A_3+B_1B_2B_3)$, and

C_4 is undetermined constant.

From eq.(20) $v_x(y)$ is a nonlinear function of y.

(ii)The mean velocity:

The mean x-velocity is determined by eq.(16) in the form

$$V^{\pm}(y)=C_1(A_1B_1)^{-1}[A_1^2\ A_3+B_1^2\ B_3][C_2^{\pm}-C_3^{\pm}\ \delta y].$$

The plus (minus) sign indicates the mean velocity of the upwords (downwords) going molecules. A more accurate result can be obtained by averaging i.e.

$$V(y)=\tfrac{1}{2}[V^+(y)+V^-(y)]=C_1(A_1B_1)^{-1}[A_1^2\ A_3+B_1^2\ B_3][\langle C_2\rangle-\langle C_3\rangle\ \delta\ y] \qquad (21)$$

Where

$\langle C_2\rangle=\langle C_2(\tfrac{1}{2})\rangle+\langle C_2(-\tfrac{1}{2})\rangle$, $\langle C_3\rangle=\langle C_3(\tfrac{1}{2})\rangle-\langle c_3(-\tfrac{1}{2})\rangle$

$\langle C_2(\tfrac{1}{2})\rangle=(B_2-\tfrac{1}{2}\ \delta)(A_1A_3)^{-1}\langle v(\tfrac{1}{2})\rangle$. etc... It is seen that V(y) is a linear function of y.

(iii)The Pressure deviatior:

The pressure deviator is defined by $P_{xy}=m\int c_xc_y f\ d\underline{c}$ or in the final form.

$$P_{xy}=C_1\{(a+b)[A_1\ A_3-B_1\ B_3)(\langle C_2\rangle-\langle C_3\rangle\ \delta\ y)+\langle C_3\rangle(A_1A_2A_3+B_1B_2B_3)]$$
$$-\gamma(b-a)/2\pi[(bA_1A_3+aB_1B_3)(\langle C_2\rangle-\langle C_3\rangle\delta y)+\langle C_3\rangle(bA_1A_2A_3-aB_1B_2B_3)]\}. \qquad (22)$$

5) Discussion and comparisons with other results:

The situation studied here is a matter of proper conditions that are imposed at the boundaries. We shall discuss the dependence of the flow velocity function, the mean velocity, and the coefficient of viscosity on the normal velocities and the degree of rarefaction.

i) The effect of nonsymmetry of the flow with respect to the line $y=0$ as a consequence of different suction velocities and diffuse reflections at the walls may be reduced to the case of symmetry by taking $a = b$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$, and let $y = 0$ in eq.(19). This gives

$$\langle C_3 \rangle = 0$$

Which means that $\langle v(\tfrac{1}{2}) \rangle = \langle v(-\tfrac{1}{2}) \rangle = \tfrac{1}{2}\varepsilon$. Therefore

$$\langle v_{x2}(0) \rangle = \langle v_{x1}(0) \rangle \tag{23}$$

as it should be expected.

ii) At the boundaries, the flow velocity rate can be obtained on one hand by subtracting eq.(13) from eq.(14). On the other hand by subtracting the equation composed of ½[eq.(7)+eq.(8)] from that composed of ½[eq.(5)+eq.(6)]. Thus comparing the two results yields

$$\partial v(\pm\tfrac{1}{2}) / \partial y = [A+B+C] L^{-1}. \tag{24}$$

Where

$y=1$, $A= 2S(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)$; $B=[(1-\varepsilon_1)\varepsilon_1-\varepsilon_2 S][(1+\varepsilon_2)+S(1-\varepsilon_2)]$

$C=[2S(1-\varepsilon_2)-\varepsilon_1][S(1+\varepsilon_1)+(1-\varepsilon_1)]$; $L=8u\sqrt{\pi}\,Kn\,S(\varepsilon_1+\varepsilon_2-\varepsilon_1\varepsilon_2)$.

As $S = 1$ i.e. at equal porosity or in the absence of it eq.(24) reduces to.

$$\partial v/\partial y=[1-\tfrac{1}{2}(\varepsilon_1+\varepsilon_2)](u\sqrt{\pi}\ Kn[(2-\varepsilon_1)/\varepsilon_1+(2-\varepsilon_2)/\varepsilon_2])^{-1} \quad (25)$$

also, the subtraction of eq.(6) from eq.(7) and eq.(5) from eq.(8) gives as S=1:

$$v^+_{x2}(\tfrac{1}{2})-v^+_{x1}(-\tfrac{1}{2})=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2). \quad (26)$$

$$v^-_{x2}(\tfrac{1}{2})-v^-_{x1}(-\tfrac{1}{2})=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2) \quad (27)$$

Which means that the relative upword and downword velocities between the walls are equal and constant. Adding up eqs.(26) and (27) gives the expression.

$$\langle v_{x2}(\tfrac{1}{2})\rangle-\langle v_{x1}(-\tfrac{1}{2})\rangle=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2)=\partial v(\pm\tfrac{1}{2})/\partial y. \quad (28)$$

Comparing eqs.(25) and (28) one finally obtains

$$\partial v_x(\pm\tfrac{1}{2})/\partial y=(1+u\sqrt{\pi}\ Kn[(2-\varepsilon_1)/\varepsilon_1+(2-\varepsilon_2)/\varepsilon_2])^{-1} \quad (29)$$

Taking into account the second factor of symmetry i.e. $\varepsilon_1=\varepsilon_2$, eq.(29) leads-in dimensional form-to the known result [Chapman & Cowling 1970].

$$\partial v_x/\partial y=U[d+2u\ \ 1\ \ (2-\Theta)/\varepsilon]^{-1}$$

When the gas is dense, Kn $\to$ 0, the flow velocity tends to the hydrodynamic limit $v_x=(y/d)U$.

(iii)Refering to eq.(19) in the absence of porosity we have

$$[\partial v(\pm y)/\partial y]y=(1-\delta)^{-1}(1-\delta[\Theta(y)-\Theta(-y)]y)\partial v(\pm\tfrac{1}{2})/\partial y,$$

by vertue of eq.(29) we get

$$[\partial v(\pm y)/\partial y]y=(1-\delta)^{-1}(1-\delta[\Theta(y)-\Theta(-y)]y)(1+u\sqrt{\pi}\,Kn[(2-\varepsilon_1)/\varepsilon_1+(2-\varepsilon_2)/\varepsilon_2])^{-1}. \quad (30)$$

Two limiting cases arise.

a)The very dilute gas Kn$\to\infty$ $(\delta\to 0)$ gives $\partial v(\pm y)/\partial y=0$ which implies that $v(\pm y)=v_o$. Where v_o is an arbitrary constant.

b) The very dense gas, the contineous gas, $Kn \to 0$ $(\delta \to \infty)$ gives from eq.(19) the flow velocity rate in either half space $y \lessgtr 0$:

$$\partial v(\pm y)/\partial y = \Theta(y) - \Theta(-y), \text{ by integrating}$$

$$v_x(y) = y + v_o$$

This shows that the flow velocity is linear. Cases a&b [see fig.(1)] agree respectively with the zeroth-approximation - the Knudsen collissionless gas- and the first approximation - the Navier Stokes equations - of the hydrodynamic equations for hard sphere model in a contineous gas derived from Boltzmann equation.

(iv)In the case of a contineous gas $\delta \to \infty$ and the absence of porosity the mean velocity amounts to $V(y)=y$. This simple relationship is similar to that predicted from the first approximation of the hydrodynamic equations or the Navier-Stokes equations.

(v)From the general expression of the mean velocity eq.(21) one can determine the slip velocity at the upper wall .

$$V_s(\tfrac{1}{2})=\tfrac{1}{2}-V(\tfrac{1}{2})$$

It is plotted against 1-the suction factor S for constant degree of rarefaction δ . It seen from fig(2a) that $V_s(\frac{1}{2})$ behaves nonlinearly as it approashes to the transition region $\delta = 1$, and linearly for both contineous and collissionless gases. 2-The degree of rarefaction δ for constant suction factor S. It is shown in fig(2b) that $V_s(\frac{1}{2})$ increases nonlinearly in the rarefied region, it sharply drops down to a minimum as

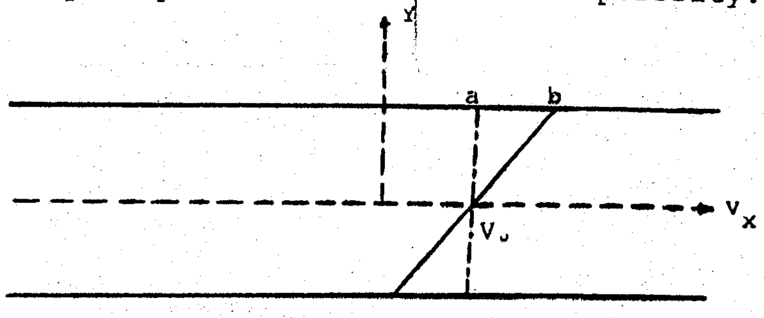

Fig.(1) a) Knudsen gas ($Kn = \infty$).
b) Navier-Stokes gas ($Kn = 0$).

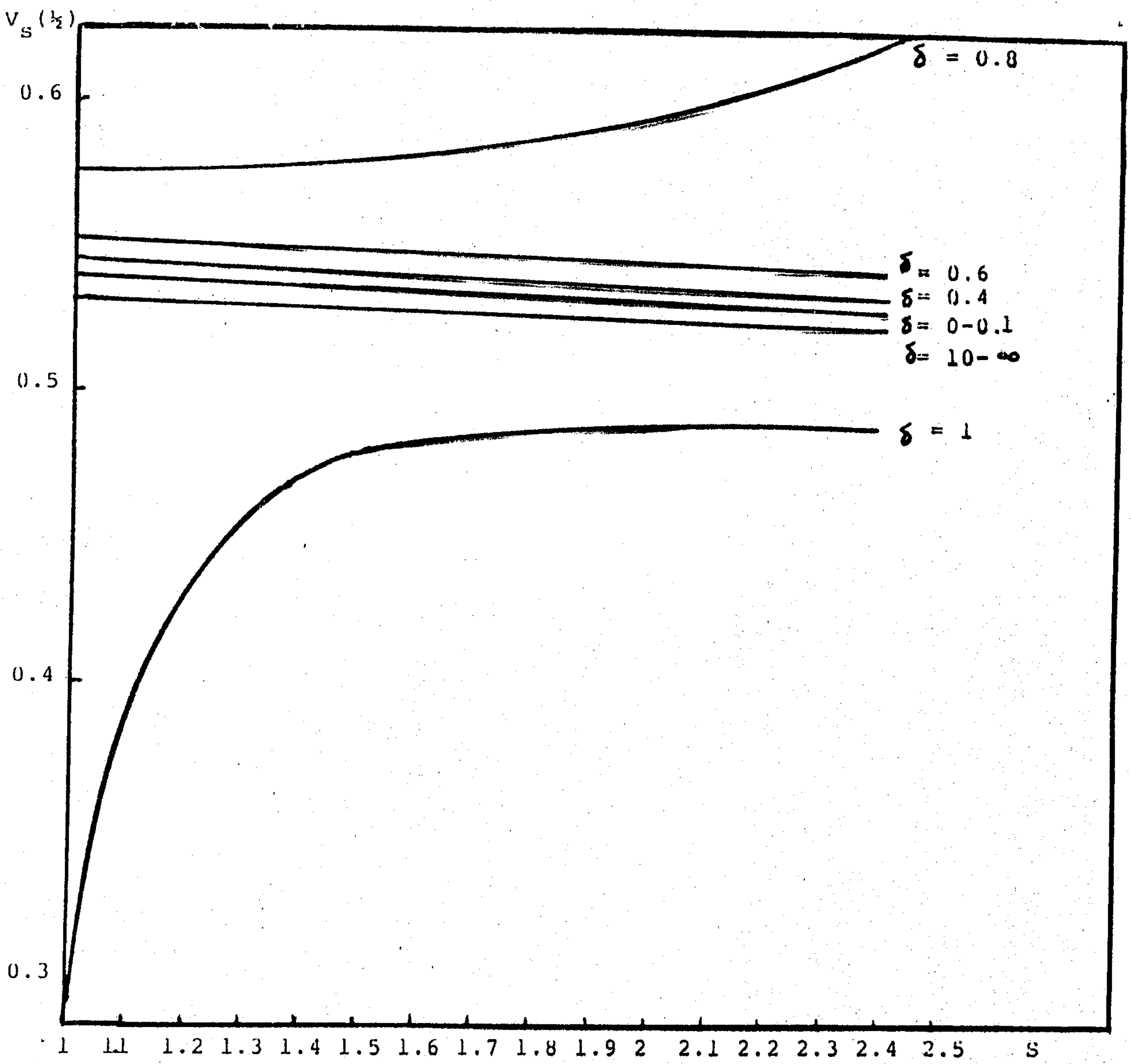

Fig.(2a) Variation between the slip velocity and suction factor for constant Mach n. $\gamma = 0.1$ and different degrees of rarefaction δ . $\epsilon_1 = 0.4$, $\epsilon_2 = 0.8$.

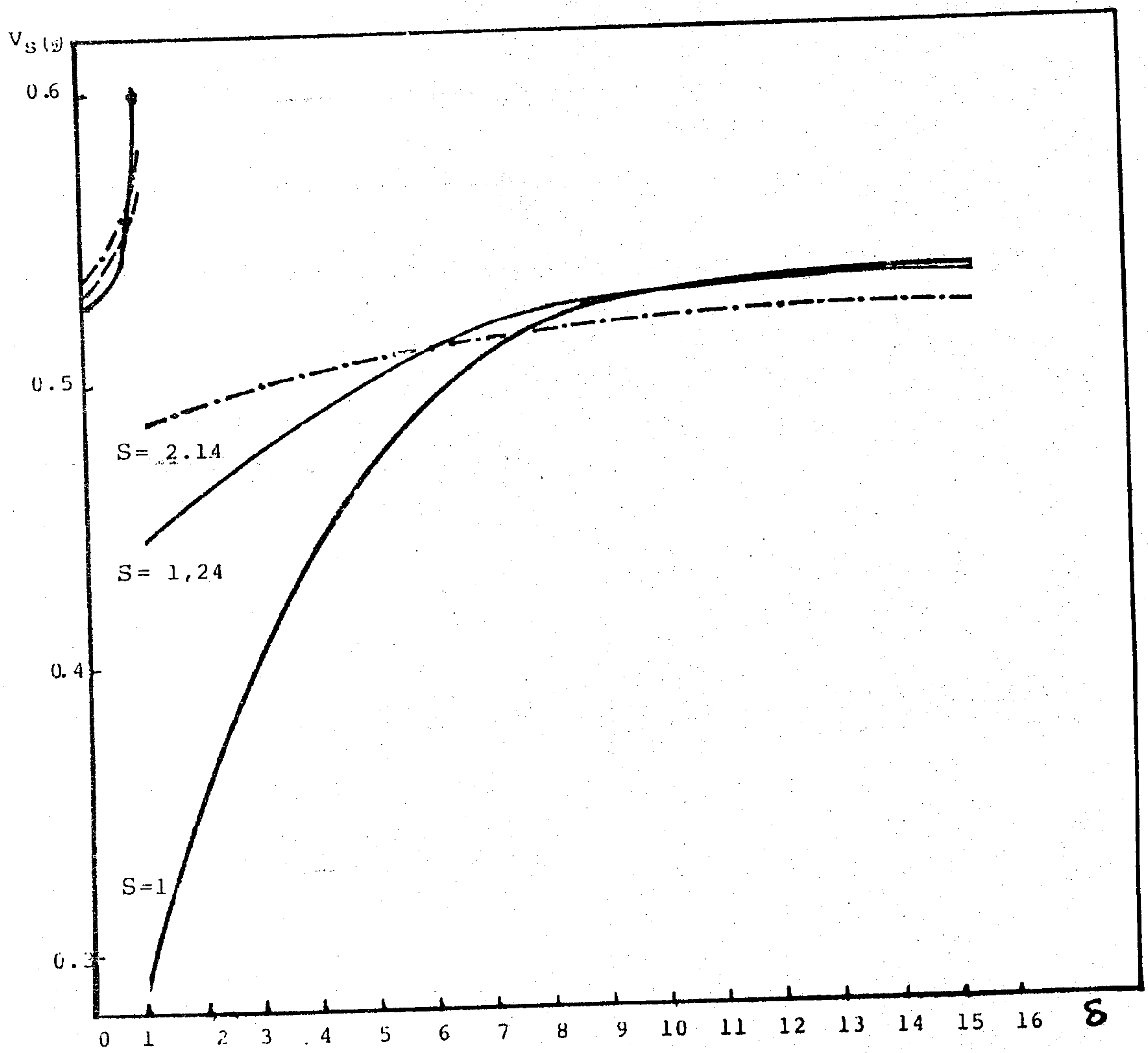

Fig.(2b) Variation between the slip velocity and the degree of rarefaction for constant Mach n. $\gamma = 0.1$ and different suction factors S. $C_1 = 0.4$, $C_2 = 0.8$.

$\delta \to 1$ and then increases to a saturation as $\delta \to \infty$. The peculiare behavior of the flow in the transition region is facing the researchers, it needs further investigations.

(vi)For nonporous walls the shear stress (22) and the coefficient of viscosity $\mu=P_{xy}(\partial v/\partial y)^{-1}$ vanish. This is consistent with the result obtained by [Mahmoud 1985].

Final Comment:

The approximate results obtained here, eqs.(19),(20),(21) and (22) give physically reasonable interpolations between Couette flow behavior for unrestricted gas density and the conditions of nonsymmetry imposed at the walls.

Of Course the easier way whould be to derive the boundary conditions and the transport equations from a kinetic picture rather than from the successive approximations of the general hydrodynamic equations of nonuniform gases.

REFERENCES

Cercignani C., "Theory and application of the Boltzmann equation" Scottish Academic Press, Edimburg, 1975.

Chapman S. and Cowling T.G., "The mathematical theory of non-uniform gases", Cambrige U.P., Cambridge 1970.

Hady F.M., M.Sc. Thesis-Assiut Unversity, Egypt. 1976.

Johnson E.A., "AERE report 10461", Her Majesty's stationery office, London, 1982.

Khidr M.A., Ph.D. Thesis-Moscow University-USSR. 1970.

Kogan M.N., "Rarefied gas dynamics," Plenum, New York 1969.

Liu C.Y., Lees L.: in Rarefied gas dynamics, edited by L. Talbot. Academic, New York, 1961.

Mahmoud M.A., Ph.D. Thesis, Menoufyia University, 1985.