MANSOURA JOURNAL OF Mathematics
Official Journal of Faculty of Science, Mansoura University, Egypt
E-mail: scimag@mans.edu.eg ISSN: 2974-4946

Stationary Coefficients with Uncertainly Under Mean Square and Mean Fourth Lyapunov Construction

M.A. Sohaly, AM.Mosaad

Department of Mathematics - Faculty of Science, Mansoura University - Egypt

Received: 3 / 1 / 2019
Accepted: 26 / 1 / 2019

Abstract: In this work, we apply lyapunov construction according to stochastic difference model where the randomness is to be white noise term and random variable stationary coefficients. Our presentation will bring dealing with mean square and mean fourth calculus. We indicate how the use of random coefficients in the difference model can provide useful insights into the qualitative behavior of the stochastic process solution according to suitable random distributions rather than the deterministic coefficients

keywords: random models; Lyapunov functional; mean square asymptotically stable; mean fourth asymptotically stable.

1.Introduction

Discrete dynamical system in deterministic and stochastic case is popular enough with many researches [1, 2, 3, 4]. The random models or random methods are very interested than in deterministic case so, in this paper, we study Lyapunov functionals construction technique in random case in order to discuss the qualitative behavior of the stochastic process solution according to some stochastic difference problems when its coefficients are stationary random variables(SRVs).

Many papers have been studied the stochastic dynamical models in continuous case such as stochastic partial differential equations or discrete such a stochastic difference equation [3,5]. Construction of Lyapunov functionals is usually used for investigation of the stability of hereditary systems which are described by functional differential or difference equations and have numerous applications [6]. The general method of Lyapunov functionals construction for stability investigation of dynamical systems was proposed and developed (see [7, 8, 9]). This latter approach allows us the consideration of a wider kind of randomness because, apart from Gaussian, other RVs like binomial, Poisson, uniform, beta, exponential, etc. can also be included in the mathematical model. Throughout this paper we propose using the mean square and fourth calculus, in order to find the norm stability conditions for the following random stationary coefficients difference model:

$$X_{i+1} = \sum_{l=-m}^{i} c_{i-l} x_l + \sum_{j=0}^{i} \sum_{l=-m}^{j} \sigma_{j-l}^{i-j} x_l \omega_{i+1} \ , \ i \in z$$

$$X_i = \phi_i \qquad (1)$$

Where ci are positive stationary random variables coefficients defined in a complete probability space (Ω,F,P), and σi are known constants , ωi is a sequence of ωi adapted mutually independent random variables with:

$$E[\omega_i] = 0, \quad E[\omega_i^2] = 1.$$

In the deterministic coefficients frame work the solution of the random model (1) has been studied qualitatively using the Lyapunov construction [10]. In this paper, we propose the mean square and fourth Lyapunov construction, inspired in its deterministic counterpart, to find the asymptotically conditions in the norm formula of the solution stochastic process according to the random model (1), that to the best of our knowledge has not been proposed yet.

The rest of this paper is given as follows: Insection.2, we prescribe some definitions and remarks. In Sections.3, our technique using for getting the mean square and mean fourth stability conditions. Finally, in section.5, we give the summary of our contribution.

The rest of this paper is given as follows: Insection.2, we prescribe some definitions and remarks. In Sections.3, our technique using for getting the mean square and mean fourth stability conditions. Finally, in section.5, we give the summary of our contribution.

Definition 1.[10]. The trivial solution of (1) for some p>0 is called:

- P-stable, p >0, if for each s >0 there exists a δ >0 such that:

$E[|x_i|^p] < \varepsilon$, $i \in Z$,

If: $\|\Phi\|^p = \sup_{i \in Z_0} E[|\Phi|^p] < \delta$ (2)

- Asymptotically p-stable if it is p-stable and for each initial function ϕi the solution xi of (1) is a symptotically p-trivial.

- **Theorem 1.[10].** Let there exist anon-negative functional Vi= V(i, x−m, ..., xi) which satisfies the conditions are the following:

- $E[V(0,\Phi_{-m},\ldots,\Phi_0)] \le c_1 \|\Phi\| p$

- $E[\Delta V_i] \le -c_2 E[|x_i^p|]$, $i \in Z$ (4)

Where c1,c2 and are positive constants. Then the trivial solution of (1) is a asymptotically p stable.

A real random variable X defined on the probabilities space (Ω,J,P) and satisfying the property that:

E[|x|p∞ > [

is called p-order random variable (p − r.v) where, p ≥1 and E[ ] denotes the expected value operator. If X ∈ Lp(Ω), then the Lpnorm is defined as:

Remark.1. If X, Y L (Ω),q ≥1 we have:

$\|XY\|q \le \|X\|2q \; \|Y\|2q$

Lyapunov's inequality:

For $1 \le r < s < \infty$, then we have:

[E[|x|]] ≤ [E[|x|]]

3. Mean Square Fourth Construction of the Lyapunov Functional

In this context, we are interested in the stability in the mean as well as the stability in mean square and mean fourth calculus. Assume that, $X(t)$ be the stochastic solution process of the stochastic difference equation (1) and we use the following notations:

$K_n = \max(k,0)$

$\alpha = \sum_{l=1}^{\infty} \| \sum_{m=l}^{\infty} c_m \|_2$

$$\alpha_1 = \sum_{l=0}^{\infty} \|c_l\| \qquad (5)$$

$S^0 = \sum_{p=0}^{\infty} \left(\sum_{l=0}^{\infty} |\sigma_l^p| \right)^2$

$S_r = \sum_{i=r}^{\infty} \sum_{j=0}^{\infty} |\sigma_j^i|, \quad r = 1,2,\ldots$

$\eta_i = \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} \; x_l \; \omega_{j+1}$ i=0,1,2,.

We begin the Lyapunov construction by assuming that, the Lyapunov functional V_{1i} has to be chosen in the form:

$V_{1i} = x^2$

Finally, we can calculate $E[\Delta V_{1i}]$ as the following:

$E[\Delta V_{1i}] = E(x_{i+1}^2 - x_i^2)$

$= E\left[\sum_{l=-m}^{i} c_{i-l} x_l + \eta_i\right]^2 - E[x_i^2]$

$= -E[x_i^2] + \sum_{k=1}^{3} T_k$

Where,

$T_1 = E\left[\sum_{l=-m}^{i} c_{i-l} \; x_l\right]^2$

$T_2 = 2E\left[\eta_i \; \sum_{l=-m}^{i} c_{i-l} \; x_l\right]$

$T_3 = E[\eta_i^2]$

For,

$T_1 = \| \sum_{l=-m}^{i} c_{i-l} \; x_l \|_2^2$

$= \sum_{l=-m}^{i} \|c_{i-l} \; x_l\|_2^2$

$\le \sum_{l=-m}^{i} \|c_{i-l}\|_4 \sum_{l=-m}^{i} \|c_{i-l}\|_4 \|x_l\|_4^2$

$\le \alpha_1 \sum_{l=-m}^{i} \|c_{i-l}\|_4 \|x_l\|_4^2$

Also,

$|T_2| =$

$2|E[\eta_i(\sum_{l=-m}^{j} c_{i-l} \; x_l + \sum_{l=j+1}^{i} c_{i-l} \; x_l)]| =$
$2\left|E\left[\sum_{j=0}^{i-1} \sum_{k=-m}^{j} \sigma_{j-k}^{i-j} x_k \omega_{j+1} \; \sum_{l=-m}^{j} c_{i-l} \; x_l\right] + E\left[\sum_{j=0}^{i-1} \sum_{k=-m}^{j} \sigma_{j-k}^{i-j} x_k \omega_{j+1} \sum_{l=j+1}^{i} c_{i-l} \; x_l\right]\right|$ (7)

Using that:

$E\left[x_k \; x_l \; \omega_{j+1}\right] = 0 \quad \forall \, k,l \le j$ (8)

Then we have:

$|T_2| =$

$2\left|E\left[\sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sigma_{j-k}^{i-j}x_k \ \ \omega_{j+1} \ \ \sum_{l=j+1}^{i}c_{i-l}x_l\right]\right|$ (9)

Then:

$$|T_2| \le 2E\left[\sum_{j=0}^{i-1}\sum_{k=-m}^{j}\left|\sigma_{j-k}^{i-j}\right|\left|x_k\omega_{j+1}\right|\sum_{j+1}^{i}|c_{i-l}||x_l|\right]$$

$$\le \sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}\left|\sigma_{j-k}^{i-j}\right|\|c_{i-l}\|_2(\|x_k\|_2^2 + \|x_l\|_2^2)$$

$$= 2E\left[\sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}|c_{i-l}|\left|\sigma_{j-k}^{i-j}\right|\left|x_k\omega_{j+1}\right||x_l|\right]$$

$$= 2E\left[\sum_{j=0}^{i}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}\left|\sigma_{j-k}^{i-j}\right||c_{i-l}|\left|x_k \quad \omega_{j+1}x_l\right|\right]$$

$$\le E\left[\sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}\left|\sigma_{j-k}^{i-j}\right||c_{i-l}|\begin{bmatrix}(x_k \ \ \omega_{j+1})^2\\ +(x_l)^2\end{bmatrix}\right]$$

$$= \sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}\left|\sigma_{j-k}^{i-j}\right| \ E\left[(c_{i-l})\left(\left(x_k\omega_{j+1}\right)^2 + x_l^2\right)\right] \le \left[\sum_{j=0}^{i-1}\sum_{k=-m}^{j}\sum_{l=j+1}^{i}\left|\sigma_{j-k}^{i-j}\right|\|c_{i-l}\|_2\left\|\left(x_k\omega_{j+}\right.\right.\right. x_l^2\left.\left.\right\|_2\right] \quad (11)$$

Since: $E\left[\left(x_k\omega_{j+1}\right)^2\right] = E[x_k^2] \ \forall \ k \ge j$ (12)

Therefore,

$$|T_2| = \sum_{l=1}^{i}\left(\|c_{i-l}\|_2\sum_{j=0}^{l-1}\sum_{k=-m}^{j}\left|\sigma_{j-k}^{i-j}\right|\right)\|x_l\|_2^2 \ + \ \sum_{k=-m}^{i-1}\left(\sum_{j=k_n}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=j+1}^{i}\|c_{i-k}\|_2\right)\|x_k\|_2^2 \quad (13)$$

Additionaly,for: $l \le I$ we have:

$$\sum_{j=0}^{l-1}\sum_{k=-m}^{j}\left|\sigma_{j-k}^{i-j}\right| \le \sum_{j=0}^{i-1}\sum_{k=-m}^{j}\left|\sigma_{j-k}^{i-j}\right| \le S_1 \quad (14)$$

And:

$$\sum_{j=k_m}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=j+1}^{i}\|c_{i-k}\|_2 = \sum_{j=k_m}^{i-1}\left|\sigma_{j-k}^{i-j}\right|\sum_{l=0}^{i-j-1}\|c_{i-k}\|_2 \le \alpha_1 \ \sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|$$

So,

$$\le S_1\sum_{l=1}^{i}\|c_{i-l}\|_2\|x_l\|_2^2 + \alpha_1\sum_{k=-m}^{i-1}\sum_{p=1}^{k_n}\left|\sigma_{i-k-p}^{p}\right| \|x_k\|_2^2$$

Hence:

$$\|T_2| \le S_1 \ \sum_{l=1}^{i}\|c_{i-l}\|_4\|x_l\|_4^2 + \alpha_1 \ \ \sum_{k=-m}^{i-1}\sum_{p=1}^{k_n}\left|\sigma_{i-k-p}^{p}\right|\|x_k\|_4^2]$$

Also,

$$|T_3| = \left|\sum_{j=0}^{i}E\left(\sum_{l=-m}^{j}\sigma_{j-l}^{i-j}x_l\right)^2\right|$$

$$= \left|\sum_{j=0}^{i}E\left[\sum_{l=-m}^{j}\left|\sigma_{j-l}^{i-j}\right|^{\frac{1}{2}}\left|\sigma_{j-l}^{i-j}\right|^{\frac{1}{2}}x_l\right]^2\right| \quad (15)$$

Then, using (10) we have:

$$|T_3| \le \sum_{j=0}^{i}\left(\sum_{l=-m}^{j}\left|\sigma_{j-l}^{i-j}\right|\right)\left(\sum_{k=-m}^{j}\left|\sigma_{j-k}^{i-j}\right| E[|x_k^2|]\right)$$

$$= \sum_{p=0}^{i}\left(\sum_{l=-m}^{i-p}\left|\sigma_{i-p-l}^{p}\right|\right)\left(\sum_{k=-m}^{i-p}\left|\sigma_{i-p-k}^{p}\right|\|x_k\|_2^2\right)$$

$$\le \sum_{p=0}^{i}\left(\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)\left(\sum_{k=-m}^{i-p}\left|\sigma_{i-p-k}^{p}\right|\|x_k\|_2^2\right)$$

$$= \sum_{k=-m}^{i}\left(\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)\|x_k\|_2^2$$

$$\le \sum_{k=-m}^{i}\left(\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)\|x_k\|_4^2 \quad (16)$$

Finally

$$E[\Delta V_{1i}] \le -\|x_i\|_4^2\sum_{k=-m}^{i}A_{ik} \ \ \|x_k\|_4^2 \quad (17)$$

Where,

$$A_{ik} = (\alpha_{1+}S_1)\|c_{i-k}\|_4 + \alpha_1\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right| + \sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_l^p\right| \quad (18)$$

Noting that, $A_{i+j,I}$ for $i \in Z$ depends on j only.So,

$$\sum_{j=0}^{\infty}A_{j+i,i} = \sum_{j=0}^{\infty}\left((\alpha_1 + S_1)\left\|c_j\right\|_2 + \alpha_1\sum_{p=1}^{j}\left|\sigma_{j-p}^{p}\right| + \sum_{p=0}^{j}\sigma_{j-p}^{p}\sum_{l=0}^{\infty}\left|\sigma_l^p\right|\right)$$

$\leq (\alpha_1 + S_1)\alpha_1 + \alpha_1 \ \sum_{j=0}^{\infty}\sum_{p=1}^{j}\left|\sigma_{j-p}^{p}\right| + \sum_{j=0}^{\infty}\sum_{p=0}^{j}\left|\sigma_{j-p}^{p}\right|\sum_{l=0}^{\infty}\left|\sigma_{l}^{p}\right|$

$\leq \alpha_1^2 + 2\alpha_1 \ S_1 \ + S_0$ (19)

Hence, we find that the following inequality:

$$\alpha_1^2 \ + 2\alpha_1 S_1 + S_0 < 1 \qquad (20)$$

is the asymptotically mean square condition for the stochastic solution of Eq.(1).

Therefore, we can write the condition (20) in the mean square form:

$$(\sum_{l=0}^{\infty}\|c_l\|_2)^2 + 2(\sum_{l=0}^{\infty}\|c_l\|_2)^2 S_1 + S_0 < 1 \qquad (21)$$

Or in mean fourth form:

$$(\sum_{l=0}^{\infty}\|c_l\|_4)^2 + 2(\sum_{l=0}^{\infty}\|c_l\|_4)^2 S_1 + S_0 < 1 \qquad (22)$$

Additionally, the main condition that the random variables we use must be bounded like as Beta distribution, Binomial distribution, etc.

Remark2. From the norm conditions (21) and (22) we note that this condition depend on the statistical distribution of the random variables c_l since, these random variables may have the same Statistical distribution or different statistical distributions (independent or non-independent) so, for only one random variable coefficient in the difference model we have many stability conditions according to their statistical distributions. Additionally, we have two norm conditions formula as in (21) and (22).

4. Conclusions

In this paper we have studied the mean square fourth calculus for the linear stochastic difference equation with white noise term random coefficients using Lyapunov functional construction technique. Sufficient conditions for the mean square and mean fourth asymptotically stable of the stochastic difference equation has been provided.

Competing interests

The author declares that there is no conflict of interests.

Acknowledgements

The authors sincerely thank the referees for constructive suggestions and corrections, which have significantly improved the contents and the exposition of this paper.

5. References

1 J.Luo,L.Shaikhet, (2007) Stability in probability of nonlinear stochastic volterra difference equations with continuous variable, Stochastic Analysis and Applications **25** (6) 1151–1165.

2 G. P. Pelyukh, Representation of solutions of difference equations with a continuous argument, Differential nye Uravneniya **32** (2) (1996)256–264.

3 L. Shaikhet, (2004)Lyapunov functionals construction for stochastic difference second-kind volterra equations with continuous time, Advances in Difference Equations 2004 (1) 262861.

4 Y. L. Maistrenko, A. Sharkovsky, (1986).Difference equations with continuous time as mathematical models of the structure emergences

5 M.A.SOHALY, (2014)Mean square convergent three and five points finite difference scheme for stochastic parabolic partial differential equations, Electronic *Journal of Mathematical Analysis and Applications* **2(1**) 164–171.

6 V. Kolmanovskii, A. Myshkis, (1992), Applied theory of functional differential equations. Kluwer Academic Publishers, Dordrecht) 4400–408.

7 L.Shaikhet, (2005)Stability of difference analogue of linear mathematical inverted pendulum, Discrete Dynamics in Nature and Society 2005 **(3**) 215–226.