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RELIABILITY MODEL FOR A LOGIC SYSTEM OF NON-IDENTICAL COMPONENTS

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ABSTRACT

The present study discusses the reliability of m-out-of-n unidentical component logic system. A mathematical model is developed to be applied for m = 1 up to n-1. The model is validated through its coincidence with the binomial distribution when all components are identical. An illustrative example is provided for the case of m=4 and n=6.

1. INTRODUCTION

In many designs, the criteria must be set to fulfill that at least m-out-of-n parallel components are good for the system to operate successfully. If all components are identical, the probability of exactly m successes-out-of-n components can be shown to obey a binomial distribution [1] with a probability density function (Pdf):

$$P = \binom{n}{m} p^m (1-p)^{n-m}, \qquad (m = 0,1,2, \ldots, n) \qquad (1)$$

where p is the success probability of any component (component reliability). Assuming constant failure rate model, the component reliability can be expressed in terms of component failure rate (λ) as [2]:

$$p = \exp[-\lambda t] \qquad (2)$$

The probability of at least m successes-out-of-n parallel components (system reliability) is given by:

$$R_p(t) = \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k} \qquad (3)$$

Practically, the degradation of components may occur with different rates. This can be attributed to different operating conditions and/ or different operational history. The present study treats this problem and considers the components to be unidentical. Under the assumption of identical components, Equation (3) will be a special case of the general model when putting all failure rates are equal.

2. MATHEMATICAL MODEL

In a parallel redundant configuration [2,3], all the n components are allowed to operate simultaneously. The system states will be denoted by 0, 1, 2, ..., n; where the 0-state expresses all components are good while the n-state expresses all components are bad. The transition probability matrix of n-unidentical component system is given by:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\ \left[\begin{array}{ccccccc} 1-\sum_{i=1}^{n}\lambda_i & \sum_{i=1}^{n}\lambda_i & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-(\sum_{i=1}^{n}\lambda_i-\lambda_2) & \sum_{i=1}^{n}\lambda_i-\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-(\sum_{i=1}^{n}\lambda_1-\lambda_2-\lambda_3) & \sum_{i=1}^{n}\lambda_1-\lambda_2-\lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1-(\sum_{i=1}^{n}\lambda_1-\lambda_2-\lambda_3-\lambda_4) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array} \right] \end{array}$$

In constructing the above matrix, the components are assumed to be non-repairable and only one failure is allowed in one transition.

If $p_o(t)$, $p_1(t)$ and $p_j(t)$ are the probability of nofailures, exactly one failure, exactly j failures respectively to occur at time t, and $p_o'(t)$, $p_1'(t)$ and $p_j'(t)$ are the corresponding first derivatives, the following set of differential equations can be obtained:

o-failure :

$$p_o'(t) = (-\sum_{i=1}^{n} \lambda_i)\ p_o(t), \tag{4.a}$$

1-failure :

$$p_1'(t) = (\sum_{i=1}^{n} \lambda_i)\ p_o(t) - (\sum_{i=1}^{n} \lambda_i - \lambda_2)\ p_1(t), \tag{4.b}$$

j-failures, 2≤ j≤n-1 :

$$p_j'(t) = (\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{j} \lambda_i)\, p_{j-1}(t) - (\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{j+1} \lambda_i)\, p_j(t). \tag{4.c}$$

n-failures :

$$p_n'(t) = \lambda_1\ p_{n-1}(t). \tag{4.d}$$

Using Laplace transform technique and applying the initial conditions $p_o(o) = 1$, and $p_j(o) = o$ for $j > 0$ [4], the solution of that set of equations can be shown to be given as :

0-failure :

$$p_o(t) = \exp[-(\sum_{i=1}^{n} \lambda_i)\ t], \tag{5.a}$$

1-failure :

$$p_1(t) = \frac{\sum_{i=1}^{n} \lambda_i}{\lambda_2} \{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2)t] - \exp[-(\sum_{i=1}^{n} \lambda_i)t]\}, \quad (5.b)$$

2- failures :

$$p_2(t) = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i - \lambda_2)\left\{ \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2 - \lambda_3)t]}{\lambda_3(\lambda_2+\lambda_3)} - \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2)t]}{\lambda_2 \lambda_3} + \frac{\exp[-(\sum_{i=1}^{n} \lambda_i)t]}{\lambda_2(\lambda_2+\lambda_3)} \right\} \quad (5.c)$$

3- failures :

$$p_3(t) = (\sum_{i=1}^{n} \lambda_i)[\prod_{k=1}^{2} (\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{k+1} \lambda_i)]\left\{ \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{4} \lambda_i)t]}{\lambda_4(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4)} + \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2 - \lambda_3)t]}{\lambda_3 \lambda_4(\lambda_2+\lambda_3)} - \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2)t]}{\lambda_2 \lambda_3(\lambda_3+\lambda_4)} - \frac{\exp[-(\sum_{i=1}^{n} \lambda_i)t]}{\lambda_2(\lambda_2+\lambda_3)(\lambda_2+\lambda_3+\lambda_4)} \right\}, \quad (5.d)$$

4- failures :

$$p_4(t) = (\sum_{i=1}^{n} \lambda_i)[\prod_{k=1}^{3} (\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{k+1} \lambda_i)]\left\{ \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{5} \lambda_i)t]}{\prod_{k=1}^{4} (\sum_{i=o}^{k-1} \lambda_{5-i})} - \right.$$

$$- \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{4} \lambda_i)t]}{\lambda_4 \lambda_5 (\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)} - \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2)t]}{\lambda_2 \lambda_3 (\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4 + \lambda_5)} +$$

$$+ \frac{\exp[-(\sum_{i=1}^{n} \lambda_i - \lambda_2 - \lambda_3)t]}{\lambda_3 \lambda_4 (\lambda_2 + \lambda_3)(\lambda_4 + \lambda_5)} + \frac{\exp[-(\sum_{i=1}^{n} \lambda_i)t]}{\prod_{k=1}^{4} (\sum_{i=1}^{k} \lambda_{i+1})} \Big\}, \qquad (5.e)$$

<u>j- failures, j> 4, odd :</u>

$$p_j(t) = (A_n)(H_j) \Big\{ \frac{\exp[-B_j t]}{(A_n - B_j)[\prod_{k=1}^{(j-1)/2} (C_{j,k} - B_j)(D_k - B_j)]} +$$

$$+ \sum_{k=1}^{(j-1)/2} \Big[\frac{\exp[-C_{j,k} t]}{[\prod_{e=1}^{(j-1)/2} (D_e - C_{j,k})] (E_{j,k} - C_{j,k})(B_j - C_{j,k})(A_n - C_{j,k})} +$$

$$+ \frac{\exp[-D_k t]}{\prod_{e=1}^{(j-1)/2} (C_{j,e} - D_k)] (F_{j,k} - D_k)(B_j - D_k)(A_n - D_k)} \Big] +$$

$$+ \frac{\exp[A_n t]}{[\prod_{k=1}^{(j-1)/2} (C_{j,e} - A_n)(D_k - A_n)] (B_j - A_n)} \Big\} \qquad (5.f)$$

<u>j - failures, j>4, even :</u>

$$p_j(t) = (A_n)(H_j) \Big\{ \frac{\exp[-B_j t]}{[\prod_{k=2}^{(j-1)/2} (C_{j,k} - B_j)(D_k - B_j)] (A_n - B_j)(G_j - B_j)} +$$

$$+ \sum_{k=1}^{(j-2)/2} \left[\frac{\exp[-C_{j,k}\, t]}{[\prod_{e=1}^{(j-2)/2} (D_e - C_{j,k})]\, (E_{j,k} - C_{j,k})\, (B_j - C_{j,k})\, (A_n - C_{j,k})\, (G_j - C_{j,k})} + \right.$$

$$+ \frac{\exp[-D_k\, t]}{[\prod_{e=1}^{(j-2)/2} (C_{j,e} - D_k)]\, (F_{j,k} - D_k)\, (B_j - D_k)\, (A_n - D_k)\, (G_j - D_k)}$$

$$+ \frac{\exp[-G_j\, t]}{[\prod_{e=1}^{(j-2)/2} (C_{j,e} - G_j)\, (D_e - G_j)]\, (A_n - G_j)\, (B_j - G_j)}$$

$$+ \frac{\exp[A_n\, t]}{[\prod_{e=1}^{(j-2)/2} (C_{j,e} - A_n)\, (D_e - A_n)]\, (G_j - A_n)\, (B_j - A_n)} \} \qquad (5.g)$$

n-failures :

$$p_n(t) = 1 - \sum_{i=0}^{n-1} p_j(t) \qquad (5.h)$$

where

$$A_n = \sum_{i=1}^{n} \lambda_i \quad , \qquad H_j = \prod_{k=1}^{j-1} \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{k+1} \lambda_i \right)$$

$$B_j = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{j+1} \lambda_i \right) \quad , \qquad C_{j,k} = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{j+1-k} \lambda_i \right)$$

$$D_k = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{k+1} \lambda_i \right) \quad , \quad E_{j,k} = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{[(j-s)/2]+k+1} \lambda_i \right)$$

$$F_{j,k} = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{[(j+s)/2]+1-k} \lambda_i \right) \quad \text{for} \quad s = \begin{cases} 1 & j \text{ odd} \\ 2 & j \text{ even} \end{cases}$$

$$G_j = \left(\sum_{i=1}^{n} \lambda_i - \sum_{i=2}^{(j/2)+1} \lambda_i \right).$$

Thus, we can find the reliability function of a system with n-unidentical components in parallel redundant configuration for each component has a different failure rate as

$$R_p(t) = \sum_{i=0}^{n-1} p_j(t) \tag{6}$$

If we let $P_a = \exp[-\lambda_a t]$ for a = 1, 2, ... , n , Equation (6) becomes

$$R_p(t) = 1 - \prod_{a=1}^{n} (1 - P_a) \tag{7}$$

Equation (7) can be recognized as a binomial process. It is therefore intuitively clear that when at least m-out-of-n components are required for the system to be in an operable state we have

$$R_p(t) = \sum_{k=m}^{n-1} \sum_{b=1}^{\binom{n}{k}} \{ [\prod_{a=1}^{k} P_{a,b}] [\prod_{a=1}^{n-k} (1 - P_{a,b})] \} + \prod_{a=1}^{n} P_a \tag{8}$$

where

$\binom{n}{k}$ is the combinational formula $\frac{n!}{(n-k)!\,k!}$.

It should be noticed the evaluation of P_a, the failure probability of "a" components, depends on which components that have been bad. The suffix b is used to assign certain set of those probablities. Shortly we can define $P_{a,b}$ as the failure probability of the b^{th} set of "a" components.

In Equation (8), if one replaces $P_{a,b}$ by P_a and $\sum_{b=1}^{\binom{n}{k}}$ by $\binom{n}{k}$, one gets

$$R_p(t) = \sum_{k=m}^{n-1} \binom{n}{k} \{ [\prod_{a=1}^{k} P_a] [\prod_{a=1}^{n-k} (1-P_a)]\} + \prod_{a=1}^{n} P_a \qquad (9)$$

if $\lambda_1 = \lambda_2 = ... = \lambda_n = \lambda$, then $P_1 = P_2 = ... = P_n = P$ in Equation (9), and equation (9) will be reduced to the same form as Equation (3) for identical components.

3. ILLUSTRATIVE EXAMPLE:

<u>4-out-of-6 unidentical components:</u>

Equations (4.a), (4.b) and (4.c) become:

<u>0- failure :</u>

$$\dot{p}_o(t) = -(\sum_{i=1}^{6} \lambda_i)\, p_o(t). \qquad (10.a)$$

<u>1- failure:</u>

$$\dot{P}_1(t) = -(\sum_{i=1}^{6} \lambda_i)\, P_o(t) - (\sum_{i=1}^{6} \lambda_i)\; P_j(t) \qquad (10.b)$$

<u>j-failures, $2 \leq j \leq 5$:</u>

$$\dot{P}_j(t) = -(\sum_{i=1}^{6} \lambda_i - \sum_{i=2}^{j} \lambda_i)\, P_{j-1}(t) - (\sum_{i=1}^{6} \lambda_i - \sum_{i=2}^{j+1} \lambda_i)\; P_j(t) \qquad (10.c)$$

with the initial conditions

$P_o(o) = 1$, $P_j(o) = o$ for $o < j \leq 5$

Taking Laplace transforms $P_j(s) = \int_0^\infty p_j(t)\, e^{-st}\, dt$, we have

$$P_o(s) = \frac{1}{(s + \sum_{i=1}^{6} \lambda_i)} \quad (11.a)$$

$$P_1(s) = \frac{(\sum_{i=1}^{6} \lambda_i)}{(s + \sum_{i=1}^{6} \lambda_i)\,[s + (\sum_{i=1}^{6} \lambda_i - \lambda_2)]} \quad (11.b)$$

and

$$P_j(s) = \frac{(\sum_{i=1}^{6} \lambda_i)\,[\prod_{k=1}^{j-1} (\sum_{i=1}^{6} \lambda_i - \sum_{i=2}^{k+1} \lambda_i)]}{(s + \sum_{i=1}^{6} \lambda_i)\,[\prod_{k=1}^{j} [s + (\sum_{i=1}^{6} \lambda_i - \sum_{i=2}^{k+1} \lambda_i)]]} \quad , \text{ for } 2 \le j \le 5 \quad (11.c)$$

To find $P_o(t)$, $P_1(t)$, and $P_j(t)$ we must take the inverse transform of the $P_o(s)$, $P_1(s)$, and $P_j(s)$.

The failure probabilities $P_i(t)$ for $o \le i \le 4$ can be easily found from Equations (5.a) - (5.e) when $n = 6$, and the failure probabilities $P_5(t)$, and $P_6(t)$ whose salution is

5- failures:

$$P_5(t) = (\sum_{i=1}^{6} \lambda_i)\,[\prod_{k=1}^{4} (\sum_{i=1}^{6} \lambda_i - \sum_{i=2}^{k+1} \lambda_i)] \quad \{ \frac{\exp[-\lambda_1 t]}{\prod_{k=1}^{5} (\sum_{i=0}^{k-1} \lambda_{6-i})} -$$

$$- \frac{\exp[-(\lambda_1+\lambda_6)\,t]}{\lambda_6\lambda_5\,[\prod_{k=2}^{4}(\sum_{i=k}^{5}\lambda_i)]} + \frac{\exp[-(\sum_{i=1}^{6}\lambda_i-\lambda_2)\,t]}{\lambda_2\lambda_3\,[\prod_{k=4}^{6}(\sum_{i=3}^{k}\lambda_i)]} +$$

$$+ \frac{\exp[-(\lambda_1+\lambda_5+\lambda_6)\,t]}{\lambda_4\lambda_5[\prod_{k=2}^{3}(\sum_{i=k}^{4}\lambda_i)](\sum_{i=5}^{6}\lambda_i)} - \frac{\exp[-(\sum_{i=1}^{6}\lambda_i-\lambda_2-\lambda_3)\,t]}{\lambda_3\lambda_4(\lambda_2+\lambda_3)[\prod_{k=5}^{6}(\sum_{i=4}^{k}\lambda_i)]} -$$

$$- \frac{\exp[-(\sum_{i=1}^{6}\lambda_i)\,t]}{\lambda_2[\prod_{k=3}^{6}(\sum_{i=2}^{k}\lambda_i)]} \Big\} , \qquad (12.a)$$

<u>6 - failures:</u>

$$P_6(t) = 1 - \sum_{i=1}^{5} P_i(t) \qquad (12.b)$$

Equations (8) becomes:

$$R_p(t) = \prod_{j=1}^{6} P_j + [\prod_{j=1}^{5} P_j](1-P_6) + [\prod_{j=1}^{4} P_j]\,P_6\,(1-P_5) + [\prod_{j=1}^{3} P_j]\,P_5\,P_6\,(1-P_4) + [\prod_{j=4}^{6} P_j]\,P_1\,P_2\,(1-P_3) + [\prod_{j=4}^{6} P_j]\,P_1\,P_3\,(1-P_2) + [\prod_{j=2}^{6} P_j]\,(1-P_1) + [\prod_{j=1}^{4} P_j]\,[\prod_{j=5}^{6}(1-P_j)] + [\prod_{j=1}^{3} P_j]\,P_5\,(1-P_6)\,(1-P_4)$$

$$+ [\prod_{j=1}^{3} P_j]\,P_6\,(1-P_5)\,(1-P_4) + [\prod_{j=1}^{2} P_j]\,[\prod_{j=4}^{5} P_j]\,(1-P_3)\,(1-P_6) +$$

$$+ [\prod_{j=1}^{2} P_j] P_4 P_6 (1-P_3)(1-P_5) + [\prod_{j=3}^{5} P_j] P_1 (1-P_2)(1-P_6) +$$

$$[\prod_{j=2}^{5} P_j](1-P_1)(1-P_6) + P_1 \ P_3 \ P_4 \ P_6 \ (1-P_2)(1-P_5)$$

$$+ [\prod_{j=2}^{4} P_j] \ P_6 (1-P_1)(1-P_5) + [\prod_{j=1}^{2} P_j][\prod_{j=5}^{6} P_j](1-P_3)(1-P_4) +$$

$$+ [\prod_{j=5}^{6} P_j] P_1 \ P_3 \ (1-P_4)(1-P_2) + [\prod_{j=2}^{3} P_j][\prod_{j=5}^{6} P_j](1-P_1)$$

$$(1-P_4) + [\prod_{j=4}^{6} P_j](1-P_2)(1-P_3)[\prod_{j=4}^{6} P_j] P_2 (1-P_1)(1-P_3) +$$

$$+ [\prod_{j=3}^{6} P_j](1-P_1)(1-P_2). \qquad (13)$$

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نموذج لعول نظام منطقى لمكونات غير متطابقة

يتضمن هذا البحث دراسة تصف عول m مكونه من مجموع n مكونه غير متطابقه (نظام منطقى) . وقد تم إظهار نموذج رياضى طبق عندما $m = 1$ حتى $m = n - 1$ وهذا النموذج محقق من خلال التطابق مع توزيع ذات الحدين عندما تكون كل المكونات متطابقة . وتم عرض مثال توضيحى فى حالة $m = 4$, $n = 6$.